ÉCOLE POLYTECHNIQUE
PROMOTION X98

RAPPORT DE STAGE D'OPTION SCIENTIFIQUE

The Geometry of Interaction and Interaction Nets in Hardware[1]

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[1]Version du 17 janvier 2003.

Résumé

La *géométrie de l'interaction* et les *réseaux d'interaction* sont deux systèmes permettant de décrire le calcul, et avec lesquels on essaye de faire des évaluateurs logiciels performants. Au cours de mon stage j'ai étudié la possibilité d'utiliser ces idées pour faire un évaluateur en hardware, afin profiter au maximum des possibilités de parallélisme qui existent en éléctronique. Mes recherches m'ont conduit successivement dans trois directions assez différentes. J'ai d'abord essayé de compiler le λ-*calcul linéaire* (qui utilise chaque variable exactement une fois) en utilisant des systèmes à base de *jetons*, inspirés de la géométrie de l'interaction. Puis, je me suis concentré sur un type de réseau d'interaction particulier, les *combinateurs de l'interaction*, qui est universel, et avec lequel j'ai cette fois essayé de compiler le λ-calcul classique, toujours en utilisant ses systèmes de jetons. Enfin, j'ai pris une approche assez différente des deux précédentes pour essayer de développer un évaluateur de réseaux d'interaction qui utilise la réduction de graphe. Un premier prototype permet de compiler le λ-calcul linéaire, et j'ai de nombreux éléments qui devraient permettre d'étendre ce prototype à un évaluateur plus universel.

Abstract

The *geometry of interaction* and *interaction nets* are two systems with which computation can be described, and that hold promise for designing high speed software evaluators. During my internship, I studied the possibility of applying these ideas to a hardware evaluator, to benifit from the inherint parallelism of electronics. My research successively followed three fairly different directions. First, I tried to compile the *linear λ-calculus* (which uses each variable exactly once) with a token passing method inspired from the geometry of interaction. Then I looked at the interaction nets formed by using the *interaction combinators*, a universal interaction system, with which I tried to compile the classical λ-calculus, again with a token based method. Finally, I tried an approach that is quite different from the two others, to try to develop an interaction net evaluator that uses graph reduction. A prototype was designed that can handle the linear λ-calculus, and I have many ideas that should allow this prototype to be extended to more general systems.

Contents

Chapter 1

Introduction

1.1 Goals

The *geometry of interaction* and *interaction nets* are two ways of describing computation, which have been developed throughout the 90s. Both systems have given rise to experimental software evaluators that try to do efficient computation. The aim of my internship was to look at ways to put these ideas into hardware, where the inherent parallelism could lead to further speed improvements. Most of my effort went into trying to find ways to encode various fragments of the λ-*calculus*, though some of my efforts can also be used on interaction nets.

From the beginning, I selected a number of criteria that give a better idea of what I was looking for, and that helped guide my work :

Evaluation time : The main goal is to evaluate terms as fast as possible, so the evaluation time of the completed circuit is a key factor.

Parallelism : The main hope in moving from software evaluation to hardware evaluation is that a degree of parallelism will become possible. Though this goal is very difficult, it is interesting to keep in mind as some strategies are better suited to being parallelized than others.

Locality : To be satisfying, an encoding of a lambda term into hardware should preserve the structure of the term. In particular, it would be desirable to have a direct correspondence between each sub-term and an area of the compiled circuit, as well as to be able to combine independently compiled sub-terms into larger terms.

Polymorphism : Much of the power of the λ-calculus comes from its polymorphism, which allows one simple term to capture an algorithm that can be applied on a whole range of types. An encoding in which polymorphism would be preserved would be very satisfying. Imagine being able to simply plug two terms together, without worrying what types they were compiled for, to get a circuit for the application of one term to the other.

Circuit size : Circuit size is of course a limiting factor from the physical point of view. However, it also impacts on compilation time since there are

optimization and placement steps that must be done before the circuit is made, which have a cost related to circuit size.

Compilation time : Since the long term goal is to be able to quickly evaluate a lambda term in hardware, it would be nice to keep the cost of compiling lower than the cost of directly evaluating the expression. A slow compiler can nevertheless be useful if the lambda term that is compiled can be evaluated with various parameters.

Though meeting all, and even some of these objectives has proven to be very difficult, they have been a great aid to evaluate the potential of a compilation strategy.

1.2 Modeling a Circuit

In order to think about compiling into hardware, it is necessary to have an idea of what is possible and what is not, what is too big, and what fits, what is too slow, and what can be done in a reasonable time. To be able to evaluate all this it is necessary to chose a model with which to work.

I chose to work in the *synchronous logic* model. It is the most widely used model today, and there are proven methods for producing circuits in it. There are also a number of languages to describe and simulate synchronous circuits. Moreover, in choosing this model I remain sufficiently far from the technical details of the fabrication process, which are irrelevant at such an early stage.

In choosing a synchronous model, I reduce the need for complex handshaking mechanisms without any major restrictions to what I can do. Moreover, the advantages that can come from using an asynchronous model should be applicable to the results I obtain here.

In restricting myself to logic circuits, I am simply clearly defining the amount of information that can flow through one wire, and avoiding having to think about factors such as noise that limit the amount of information in analog machines. In any case, whatever can be computed with an analog machine can be computed just as well with a digital machine as long as one is willing to use more wires.

1.3 The Languages that were used

In order to describe synchronous circuits, a large number of languages are available. After a survey of existing free software, I was left hesitating between *VHDL*, *Verilog* and *Esterel*. The first two are industry standards, VHDL being of a more European flavor. A number of free tools exist for them, but none seem to be equipped with all the features yet. Esterel is mostly used for circuit verification, but is shipped with an easily usable and expandable simulator. It has also been around for quite a while and is well established.

For my project, I finally opted for Esterel, because of its ease of use. All the circuits that I generated were described using this language. I later found that *Lustre* might have been a slightly better choice than Esterel because of a more convenient type system.

The other language that I used extensively was *Objective Caml*. I wrote a number of circuit generation variants, all built around a common λ-calculus parser.

1.4 A brief tour of this Document

Since the topic I had to study was quite vast, I approached it from a number of directions during my three-month internship. In the sequel, I will cover the three main approaches, each of which occupied roughly equivalent amounts of time.

For the first few weeks, I tried to compile a simplified λ-calculus by using token passing methods. Chapter 2 covers this simplified case. During this period, I learned the basics of the geometry of interaction and of interaction nets. When I read [MP01], which describes how to encode the full λ-calculus using the very simple interaction combinators, I decided to try its approach in hardware. The details can be found in chapter 3. Finally, in chapter 4, I present what I think is my most promising work, a processor for interaction nets, which uses graph reduction to carry out evaluation.

1.5 Acknowledgments

I would like to warmly thank Ian Mackie for offering me this internship at such short notice, spending hours in his office discussing ideas with me, his positive and encouraging attitude, and help in trying to weed this report of all the errors that I put into it. It has been a pleasure having him as a supervisor. I would also like to thank all the people whom I met at LIX for the pleasant atmosphere around the lab.

Chapter 2

The Linear Case

2.1 The Linear λ-calculus

In what follows, I will often be making reference to the *linear λ-calculus*. I will not be referring to the full linear λ-calculus that comes from full linear logic through the Curry-Howard isomorphism, but rather to a simplified version that comes from the exponential-free multiplicative fragment of linear logic, or from classical logic without weakening or contraction rules.

The terms of the linear λ-calculus are in fact terms from the classical λ-calculus in which each variable is used exactly once.

An example of a linear term is :

$$(\lambda xy.yx)(\lambda x.x)$$

The following well known terms are not linear because they do not use a variable (2.1) or because they use one twice (2.2).

$$\lambda xy.y \tag{2.1}$$

$$\lambda xy.x(xy) \tag{2.2}$$

The linear lambda calculus is not very interesting as a programming language because it is impossible to create or erase information in it, and consequently it can only express trivialities. Nevertheless, it is a good starting point because it is contained in the classical λ-calculus, and is in itself quite hard to compile into hardware in a satisfactory way.

2.2 Encoding the Linear λ-calculus with Linear Logic

There is a strong link between computation and logic. Indeed, the *Curry-Howard isomorphism* applied to classical logic shows that a lambda expression and a logical proof share a common mathematical structure. According to this isomorphism, doing cut elimination on a proof is identical to doing β-reduction on a lambda expression.

The Curry-Howard isomorphism relates in the same way the linear λ-calculus and the multiplicative fragment of linear logic (made up with $\otimes$ and $\parr$ connectives). I will not go into the details of the isomorphism or of linear logic, which are not in the scope of my project. The curious reader can refer to [Gir87, Gir95, Abr93] for more information. I will just give an example of how to translate a λ-expression into a linear logic proof.

Formula 2.3 shows the translation of $(\lambda x.\lambda y.xy)$ into a linear logic proof. The actual formulas that appear in the proof are not a part of the proof itself. We could use the same proof to prove other formulas by replacing a or b by arbitrary formulas. In fact, in the Curry-Howard isomorphism, the formulas of the proof representation are linked with the types of the λ-calculus representation. If we write the conclusion $(a \otimes b^\perp) \parr a^\perp \parr b$ of this proof using the linear implication connective[1], we get $((a \multimap b) \multimap a) \multimap b$, which nicely matches the type $(a \to b) \to a \to b$ of the translated lambda term.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a^\perp \quad \mathbf{a}}\,axiom \quad \dfrac{}{a \otimes b^\perp \quad a^\perp \parr b}\,axiom \quad \dfrac{}{\mathbf{b}^\perp \quad b}\,axiom}{a^\perp \quad a \otimes b^\perp \quad \mathbf{a}^\perp \parr \mathbf{b} \quad \mathbf{a} \otimes \mathbf{b}^\perp \quad b}\,tensor}{\mathbf{a} \otimes \mathbf{b}^\perp \quad \mathbf{a}^\perp \quad b}\,cut}{(\mathbf{a} \otimes \mathbf{b}^\perp) \parr \mathbf{a}^\perp \quad \mathbf{b}}\,par}{((a \otimes b^\perp) \parr a^\perp) \parr b}\,par \tag{2.3}$$

The proof notation is unfortunately weighed down by a lot of syntax. In the case of the multiplicative fragment of linear logic, a proof can be described, as is shown in [Gir87, Laf95], by a graph representation that does away with a lot of the superfluous information. These graphs are called proof nets. Figure 2.1 shows the previous example as a proof net, as well as normal graph representation of a linear λ-term. To convert between the two representations, all that needs to be done is to connect variables to their abstractions, and do the substitutions shown in figure 2.2.

In the sequel, when describing net representations, I will refer to the left, right and principal ports of the $\otimes$ and $\parr$ agents, when describing proof nets. The meanings of these words are self-explanatory.

In the graph net representation, computation is done by doing cut elimination steps. There are two cut elimination rules in the multiplicative fragment of linear logic. The first eliminates an axiom with a corresponding cut :

$$\dfrac{\vdots\ \mathbf{A} \quad \dfrac{}{\mathbf{A}^\perp \quad A}\,axiom}{A}\,cut \quad \rightsquigarrow \quad \vdots\ A \tag{2.4}$$

The second one moves a cut up one step in the proof through a $\otimes$ and $\parr$.

$$\dfrac{\dfrac{\vdots\ A \quad \vdots\ B}{A \parr B} \quad \dfrac{\vdots\ A^\perp \quad \vdots\ B^\perp}{A^\perp \otimes B^\perp}}{}\,cut \quad \rightsquigarrow \quad \dfrac{\vdots\ A \quad \vdots\ A^\perp}{}\,cut \quad \dfrac{\vdots\ B \quad \vdots\ B^\perp}{}\,cut \tag{2.5}$$

[1] Using $a \multimap b = a^\perp \parr b$, $(a \otimes b)^\perp = a^\perp \parr b^\perp$ and $(a \parr b)^\perp = a^\perp \otimes b^\perp$

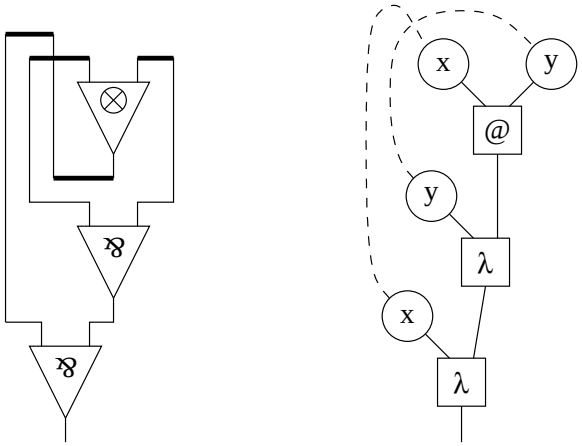

Figure 2.1: The graph on the left is the proof net for $(\lambda x.\lambda y.xy)$. Axioms and cuts have been represented by thicker lines, but this distinction isn't usually made since an axiom can be recognized by its $\sqcap$ shape, and a cut by its $\sqcup$ shape. Its structure reflects the structure of 2.3 but we no longer need to bother with the formulas. The graph on the right is an upside-down version of the standard graph representation of the λ-term. Dotted lines have been added between matching variables to show the likeness of the two graphs.

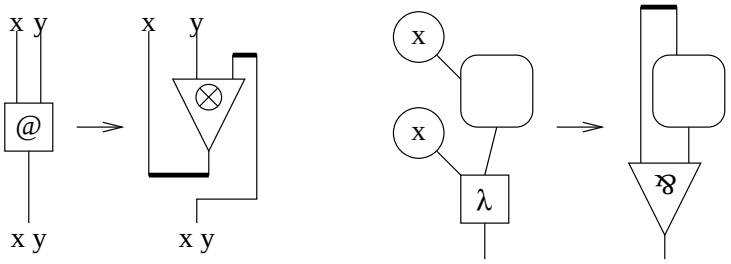

Figure 2.2: The two rules to convert an upside-down graph representation of a lambda term into a proof net.

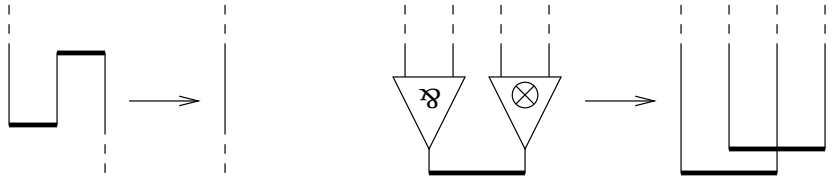

Figure 2.3: Cut elimination in proof nets is very visual thanks to the electrical analogy. It happens by eliminating zig-zags in the wires, and getting rid of unnecessary multiplexer-demultiplexer pairs.

Cut elimination in the proof net representation happens very naturally using Girard's electrical analogy [2]. Connections between connectives, cuts and axioms are all seen as wires. If two wires are plugged into each other they behave just like a single wire. In this way the cut elimination rule in 2.4 can just be seen as straightening a wire. The second rule also has an interesting electrical interpretation. If we consider $\parr$ and $\otimes$ to be a sort of multiplexer and demultiplexer, then a direct connection between a multiplexer and a demultiplexer can be reduced to a direct connection between corresponding wires. This is exactly what happens in 2.5. Figure 2.3 shows the two rules for proof nets.

If we consider two proof nets to be identical if they differ only by application of 2.4, then we find that rule 2.5 applied to a proof net has exactly the same result as β-reduction applied to the corresponding lambda term. Figure 2.4 shows the correspondence.

Now that we have seen how a λ-term can be transformed into a proof net, and how reduction of the λ-term is the same as cut-elimination on the proof net, we will be able to apply proof net methods to do computation. The next section will introduce the geometry of interaction that allows us to evaluate proof-nets.

2.3 The Geometry of Interaction Interpretation

The *geometry of interaction* is a way to assign a meaning to a net by assigning weights to its edges, which was originally developed to give a semantics of linear logic. More information can be obtained from [Gir89].

The values of the weights belong to an algebra that depends on the meaning that we wish to give to parts of the graph. The weight of a path is the product of the weights of its edges (the last edge in the path is at the leftmost position in the product). A weight can be calculated between two nodes by adding the weights of all the paths that go from one node to the other.

The graph that is considered by the geometry of interaction isn't exactly the net that we have seen above, but rather an unfolded version, in which a node is split in two parts, one for tokens going up and one for tokens going down. The edges of the graph are also appropriately split into two directed edges, one going up, and one going down. Axioms and cut links make it possible to go back and forth between the up and down directions. Another way of obtaining the same result is to keep the original nets, but to disregard paths that turn around in a connective.

[2] See [Gir95].

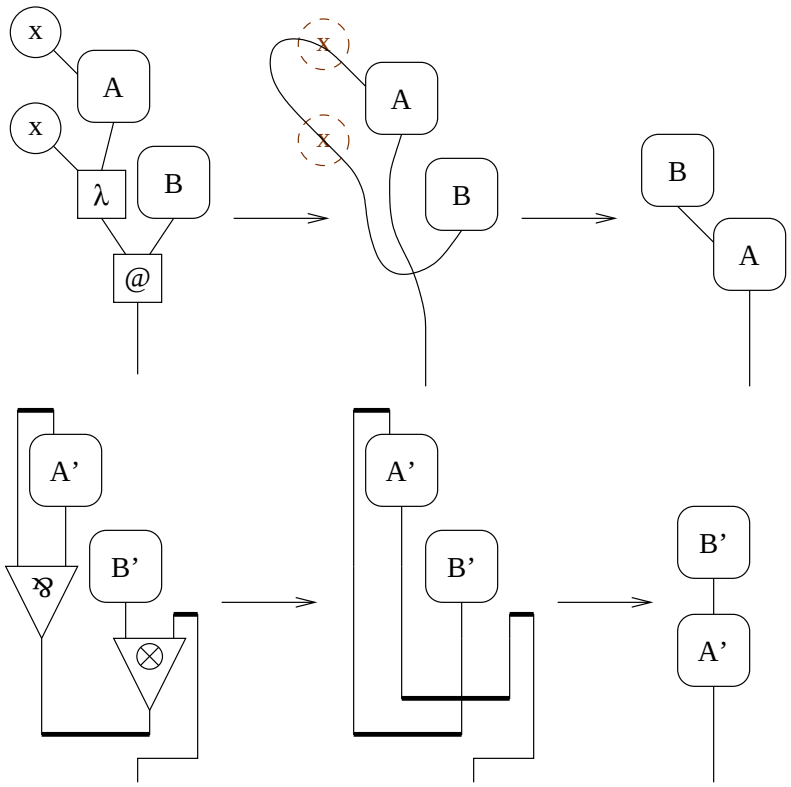

Figure 2.4: Comparison between beta reduction (top row) and cut-elimination (bottom row).

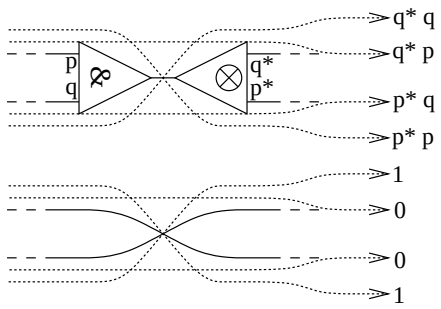

Figure 2.5: The two nets above are equivalent according to rule 2.5. For the geometry of interaction to give them identical meanings, the weights of the four possible paths from left to right must have the same weights in both cases. Equations 2.6 come directly from equating the weights of corresponding paths.

In the cases that we will be considering[3], if we start building a path from the root of the net, there is, at any given node, only one direction in which to extend the path to have a non-zero weight. This means that evaluation using the geometry of interaction can be done by moving a token around the net, keeping track of the weight that it is carrying.

For the multiplicative fragment of linear logic, the algebra that is used is particularly simple, as it is generated by two elements p and q. The path that goes from the right (resp. left) side of a $\otimes$ or $\parr$ connective to the principal port has the weight p (resp. q). The paths going in the opposite direction have the inverse weights : p^* (resp. q^*) to go from the principal port to the right (resp. left) port.

Figure 2.5 shows how the $\otimes - \parr$ rule defines some key identities[4], which are :

$$p^*p = 1,\ q^*q = 1,\ p^*q = 0,\ q^*p = 0 \tag{2.6}$$

For the paths that we will be considering, and after reduction by 2.6, the weight will always be written as a product of ps and qs. It will never be necessary to use p^* and q^*. Thus a stack is an appropriate model for the algebra. Left-multiplying by p or q is like pushing a value onto the stack. When entering the principal port of a $\otimes$ or $\parr$ agent, the value on the top of the stack (on the left of the product) will cancel out with the weight of one of the paths, and will be zero with the other path, so we will pop the top element off the stack, and go in the direction that doesn't leave us with a zero weight. Figure 2.6 illustrates how this works.

2.4 Possibility of Parallel Evaluation

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[3]Nets built from a lambda expression returning a value of base type. If the returned value is not of base type, as is the case for a function, there are a number of non zero-weighted paths that correspond to the different parts of evaluating the function.

[4]These identities arise from our desire to have the same weight for two wires as for a configuration where $\otimes$ and $\parr$ are connected by their principal ports.

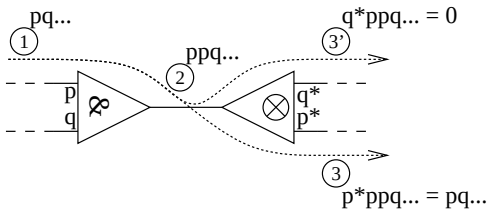


Figure 2.6: A token arrives at (1) with $pq\ldots$ at the top of its stack. It goes through the ⅋ connective, gaining a p on the top of its stack, and ending up at (2). The token can then in principle move to (3) or (3'). However, if it moves to (3') then it ends up with a zero weight. Only by going to (3) will the token have a non-zero weight. The weight of the token that exits in (3) is $pq\ldots$ just as it was initially. Everything happened as if p was pushed onto the stack between (1) and (2) and then popped off in (2) to choose between (3) and (3').

In the various encodings that I have studied in this chapter, I have considered that tokens leave from extremities (the root of the term, or a constant). This excludes the possibility of evaluating bits of path and then multiplying the results together to get the weight of the full path. In order to achieve that type of parallelism, it would be necessary for two arbitrary connectives to be able to communicate directly. In chapter 4 I will study a method that allows such communication.

For now the potential source of parallelism comes from constant binary operators, such as addition. My hope in this chapter was to find an encoding that would allow many tokens to flow through the circuit at once. At the beginning there would be a single token, at the root of the term. When this token would meet an adder (for example) it would split into two tokens. At some point these tokens would return to the adder carrying an integer. After the rendez-vous, one token would emerge with the result of the addition and would flow back to the top of the term.

In an optimized version of the above, tokens would start at all the constants and would flow around the term, coalescing at adders (for example) until only one token would be left to reach the top of the term. This simplification works because the paths of base type in the geometry of interaction are symmetrical, starting at the root, going all the way down to the constants, and then following the reverse path back up to the root.

2.5 Handling Values of Base Type

Up to now, we have only discussed compiling pure lambda expressions. In practice, it will usually be desirable to have constants for integers, booleans, and basic operators such as incrementation, decrementation, and the conditional[5].

A constant value (integer or boolean) will be inserted in the logic net as

[5]The conditional takes one boolean parameter that allows it to select one of its two other parameters. For the linear lambda calculus I chose to have a conditional operator that could only handle values of base type (integers or booleans) otherwise it would have been necessary to use parts of the additive fragment of linear logic.

an agent with one port. For the geometry of interaction, a token entering that port will immediately exit by the same port, the weight associated with this maneuver is a constant-specific element of the algebra.

A constant function will be added to the net as an agent with one output port and a number of input ports. For a unary operator, a token incoming on the output leaves by the input unchanged (multiplied by a unity weight). When it later arrives at the input, it leaves by the output with for weight an element of the algebra that operates like the function on the algebra element that represents a constant value.

When we look at typing information, we see that there is no need for the algebra element that represents a constant value to commute with p and q. Indeed, when a constant value, or a function that produces a constant value is reached, in a net constructed from a λ-term of base type, the product of the traversed weights will always be unity. Because of this, only one constant value will be carried at a time, so the algebra can be modeled as before by a stack, except that a register has to be added to store the current base value if there is one.

2.6 Information at Token Approaches

The geometry of interaction interpretation strongly suggests the possibility of an electronic implementation of linear lambda expression evaluation. In this implementation each connective of the expression we wish to compile corresponds to a physical location in the electronic circuit. A token is able to flow between connected connectives, carried by one or many wires, following the rules of the geometry of interaction.

The major problem with this approach is that the information that is carried by the token is an element of the algebra, which is infinite. In a software implementation, large amounts of memory can be allocated on the heap to store the required information. Electronically, it would of course also be possible to have a heap that the token could access. However, this would make it impossible to have many tokens flowing around the term in parallel since they would have to take turns accessing the heap. It is therefore desirable for a token to be able to carry all the information it needs along with it.

2.6.1 Mastering the Amount of Information by Typing

Luckily, in the case of the linear λ-calculus, it is possible to know at compile-time how much information the token is going to be carrying around with it at a given point in the circuit, if the type of the term is known[6]. We will find that the amount of information is closely related to the type of the term that has its root at the current point in the circuit.

The type of the term can be seen as a binary tree, with a node for each $\rightarrow$[7], and a leaf for each base type. Here are some properties of interest :

1. When the token arrives at a point in the circuit, if we start at the root

[6]For a term of polymorphic type, it is necessary to specialize the type variables.

[7]Or rather for each $\multimap$ since the function is linear, but I will dispense with this detail in notation since there is no ambiguity.

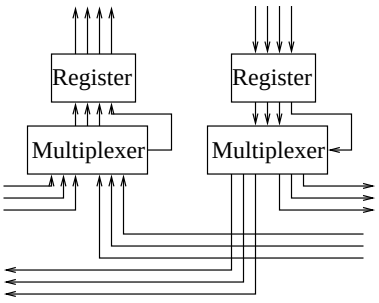


Figure 2.7: A parallel connective.

of the type tree, and descend into the tree using the values on the stack (going left for a p and right for a q) we will exactly end up at a leaf.

2. Moreover, if we annotate the base types at the leaves of the tree with + and - to show whether they are results or parameters[8] respectively, we will find that tokens going up (away from the root of the term) carry a constant value only on parameter leaves, while tokens going down (towards the root of the term) only carry one on result leaves.

3. Finally, for linear λ-terms without strange functions such as the conditional, each leaf of the type tree will be traversed exactly twice, once in each direction.

The above properties can be proven by structural induction on the λ-term.

The consequences are quite valuable for us since they allow us to know exactly how much information can flow through each point in the circuit. This will enable a number of encodings that we will now study.

2.6.2 Parallel Encoding

In the parallel encoding a token is carried at a given point of the circuit by the number of wires needed to carry the deepest stack that the token can be faced with, as well as a wire to indicate the token's presence and wires to carry the current constant value. Connectives can then be simply implemented with a multiplexer and a demultiplexer and some registers as in figure 2.7. The registers are necessary because the token is likely to flow through the same connective many times, though careful optimization could make some registers redundant.

A variant of the parallel encoding can be obtained by numbering the states that the token can be in at a given point in the circuit. The token can then be represented by the minimum number of wires that is needed to encode the

[8] In a type such as $a \to (b \to c)$, a and b are clearly parameters, while c is a result. To generalize this intuitive idea, simply count the number of times you go left in the type tree. Results have turned left an even number of times, while parameters have turned left an odd number of times. If one were to replace $A \to B$ with $\overline{A} \vee B$, then one would find that types that after application of DeMorgan's laws, types that are not complemented are results, while complemented types are parameters.

number of states. In general this approach will use fewer wires, but this advantage will be completely over-balanced by the increase in complexity of the connectives.

2.6.3 Serial Encoding

This encoding tries to reduce the size of the circuit compared with the parallel encoding. Instead of sending the token's information in parallel, we now send in a bit-serial manner, with the top of stack first. In this way the first bit of the token that arrives is able to direct the token in the right direction when there are two paths to choose from.

With this approach it is necessary to add delays on some of the links to ensure that the first bit of the token will never run into the body of the token as it loops around the graph of the term. These delays mean that the serial encoding will be slower than the parallel encoding.

The length of the delays that have to be added on links depend on the typing information, and therefore on the global structure of the term. I tried a number of methods to get rid of the delays, in order to make a serial encoding that can be built locally from the proof net of the term.

Some of my most promising attempts involved misrouting a blocked token towards the tail of the impeding token, in order to keep the information flowing, and to avoid deadlock. Unfortunately I was able to find single-token deadlock scenarios for each scheme that I came up with.

2.6.4 One-Hot[9] Encoding

This encoding is by far the simplest electronically. When considered on paper it seems to produce huge circuits, but in fact the whole circuit simply reduces to the various constants that appear in the term, connected by direct links.

The idea is that at a given point in the circuit there will be one wire for each possible token state. Since a token will only go through a given link in a given direction with a given state once during evaluation, it is no longer necessary to delay the token at the connectives. Moreover since a token arriving at a connective in a given state will always leave the connective in a predetermined direction and state, the connectives will just be wirings between incoming wires and outgoing wires.

The number of wires that interconnect connectives, can grow exponentially with the size of the term that is being compiled. Consider for example :

$$(\lambda x.x)(\lambda x.x)\ldots(\lambda x.x)$$

The leftmost $(\lambda x.x)$ will be traversed a number of times that grows exponentially with the size of the term.

Fortunately, after simplification of the wires using the connections that are made in the connectives (something that is always done before producing a circuit on silicon), the number of wires in this example reduces to one.

If there are constants in the circuit the number of wires will be the number of ports of the constants plus the size of the type of the full term. All the

[9]The term *one-hot* is used when making finite state machines in which each state has its own register.

structure of the lambda term will have disappeared, and only the operations on base types will remain. In a way this is an optimal encoding of the linear lambda term.

2.6.5 Casting from one Encoding to Another

The encodings that I have described are in fact not incompatible. It would be perfectly possible to mix terms compiled with two different approaches by using simple casting circuits.

Table 2.1 shows the central element in a converter from one encoding to another.

	To Parallel	**To Serial**	**To One Shot**
From Parallel	Nothing	Shift Register	Demultiplexer
From Serial	Shift Register	Nothing	State machine
From One Shot	Priority Encoder	Go Via Parallel	Nothing

Table 2.1: This table gives a rough idea of how to cast from one encoding to another.

Most lambda expressions have free type variables in their type. At compile time we have to specify these free variables in order to determine the maximum stack depth in the circuit. Thus $\lambda x.x$ won't compile the same way if it is typed as $int \rightarrow int$ or as $(int \rightarrow int) \rightarrow int \rightarrow int$. A more elegant possibility is to compile $\lambda x.x$ with type $unit \rightarrow unit$, which will give a circuit of minimum size. The resulting circuit can the be interfaced to a circuit that needs a larger type through a caster that stores the extra type information when the token enters the term, and restores it when leaving the term.

2.6.6 Limits of the Information at Token Approach

Now that I have given three encodings of the linear lambda calculus into hardware, and shown how casting can allow arbitrary mixing between encodings, it is time to return to the goals that I outlined in section 1.1, to see how well they have been met.

Evaluation time : In the one-hot encoding, the circuit is reduced to the necessary operators, connected by wires. Things can't get much better than that. With the parallel encoding, there is an added cost that is related to how much the information has to flow around the circuit between operators. In the serial encoding the cost is even greater as it increases with the amount of information that has to be moved.

Parallelism : Since the token will only flow through a given point in the circuit in a given state once, the one-hot encoding allows many tokens to be in the circuit, as long as the constant functions can't cause collisions[10]. This means that with a one-hot encoding, many calculations can be carried

[10]The conditional can cause collisions, for instance if two tokens are leaving the *then* and *else* branches simultaneously.

out in a pipelined way[11]. Another possibility is to do one calculation at a time, but have tokens split in two at binary operators, as described in section 2.4. For the serial and parallel encodings only one token can be at a point in the circuit at a time, so parallelism is out of the question. The same thing is true if casting is used, since the cast logic can only store the information for one token at a time, and thus only one token can be present under the caster.

Locality : All the encodings have a direct correspondence between the topology of the circuit, and the topology of the graph of the lambda term. In the one-hot encoding, however, since most of the circuit is just wiring, optimization will tend to break the correspondence.

Polymorphism : None of the encodings are directly polymorphic, though they all produce circuits that can be used on smaller types[12] than they were compiled for. With casting, terms can be used in a fully polymorphic way.

Circuit size : Unless we consider that large entities such as adders should only appear once in the circuit, the one-hot encoding is as compact as can be expected. The other two encodings are somewhat larger since they contain logic that regulates the flow of the token around the circuit. The serial encoding is smaller than the parallel encoding.

Compilation time : The three encodings rely heavily on typing information. Since types of a sub-term can have a size that is exponential with the size of the full term, compilation time can be exponential. This is unfortunate considering that the linear lambda calculus can be reduced in a linear number of steps by simple graph reduction[13]. Because of this these encodings can only be useful if the resulting circuit is to be used many times.

It is interesting to note that for a pure linear λ-term (without constants), the most general type of the term directly leads to the one-hot encoding, all one needs to do is connect corresponding type variables. This means that in looking for the type, we have in fact evaluated the linear λ-term.

Globally, the various encodings are not very satisfying, even for something as simple as the linear λ caluclus. This encouraged me to try other approaches.

2.7 Other Token Approaches

Since the major problem with the approaches seen so far is that the token has to carry all its information around with it, I tried exploring other possibilities. Some of these possibilities will be used in later chapters.

Information at Origin : With this type of approach, the stack elements are left wherever they were created. The idea is that this will prevent the

[11] This is interesting if the constant values can be changed for each successive token to do the same calculation on many different imput values.

[12] A smaller type is a type whose tree is included the tree of the larger type.

[13] It should be noted that an encoding very close to one-hot can be obtained in linear time by using graph reduction.

token from carrying large amounts of information into small subterms where they won't be needed. By some mechanism, the token keeps track of where the information is stored, in order to fetch it when necessary. One promising approach was for each connective that is traversed by the token to remember where the token came from. In this way if the token needs to know which way to go it can simply ask behind it. Since the path that leads back to the information can get very long and can go through the same parts of the circuit many times, it is necessary to reduce loops that appear in the remembered path, when they contain no information. This method remains quite complicated, and there are always cases in which it is necessary to remember a path that goes through the same part of the circuit many times; It is not yet clear exactly what determines the maximum number of tims.

Global Information : Storing the information at a global location eliminates the need to carry anything around with the token. However all hope of parallelism vanishes. This approach is used in chapter 3.

Interaction Based : Instead of using the geometry of interaction, we can try to somehow do graph reduction directly. This is what is developed in chapter 4.

2.8 Adding Erasing

In this chapter we have entirely dealt with the linear λ-calculus where each variable is used exactly once. It is in fact possible to extend all the results of this chapter to the linear λ-calculus with erasing, in which variables are used at most once. Indeed, unused variables will just correspond to parts of the circuit that are never reached by a token.

If there was some sort of parallelism in the circuit, that came, for instance, from starting tokens at all the constants at the beginning of evaluation, the situation would be more difficult. Indeed, tokens that are useless because they end up in unused variables would uselessly move around the circuit slowing down tokens that are useful.

Chapter 3

Interaction Combinators

3.1 Interaction Nets

Interaction nets were introduced by Lafont in [Laf95]. They generalize the proof nets of chapter 2. Their strong confluence properties make them particularly interesting to work with. Because of our knowledge of proof nets, I will give as a foot-note the proof net element that corresponds to each element of an interaction net.

The first ingredient we need is a set of *agents*[1]. Each agent has one *principal port* and zero or more *auxiliary ports*[2]. An *interaction net* is made up of agents, some of whose ports have been connected by wires[3]. There can be *free ports* that are not connected to another port. There can also be wires that are not connected to any agent. They provide the net with two extra free ports each. Loops of wire are also a possibility, they of course provide no free port. The *interface* of an interaction net is made up all its free ports.

The above definition of an interaction net is completely static. We still do not know how to compute. Computation is done by adding a set of rules, so that we get an *interaction system*. A rule transforms a net made up of two agents and a wire connecting their principal ports into another net that has the same interface[4]. Agents connected by their principal ports make up an *active pair*. There can only be one rule per pair of agents, and rules that apply to a pair of identical agents must be symmetrical. In this way there is at most one rule that can be applied to an active pair, and exactly one way to apply it. Reduction of an interaction net proceeds by applying any rule that can be applied until no active pairs remain. Since agents only have one principal port each, and since rules always operate on active pairs, there can be no interference between simultaneously applicable rules. Therefore the order of application of rules is of no importance. This gives a rough idea of where strong confluence comes from. For the details, refer to [Laf95].

[1] $\otimes$ and $\parr$ in proof nets

[2] For $\otimes$ and $\parr$, the inferred port is the principal port, the other two ports are auxiliary ports.

[3] Just as the wires of proof nets connect agents. There is no longer any need for axioms or cuts.

[4] The $\otimes$-$\parr$ cut-elimination step replaces two agents connected by their principal ports into two wires. The number of auxiliary ports before and after is four.

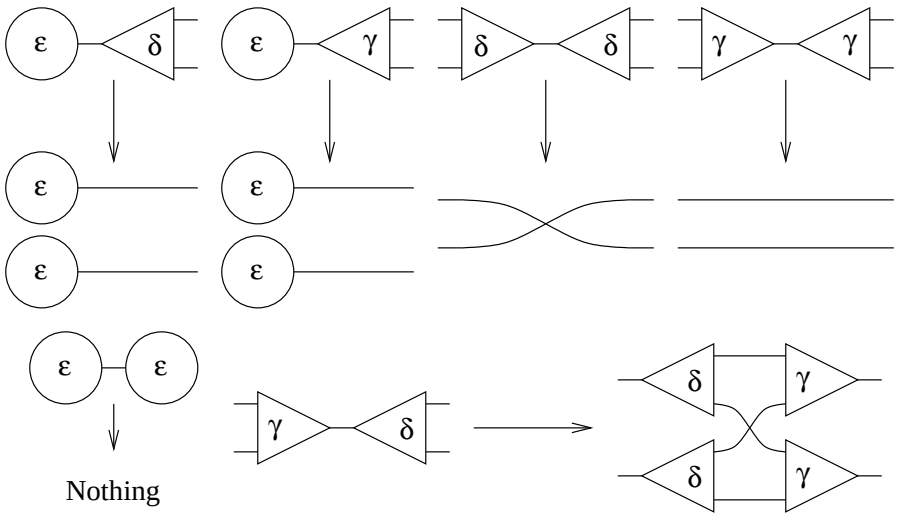


Figure 3.1: The interaction combinators reduce according to six very simple rules.

In the rest of this chapter, we will be using one particular interaction system, that will be introduced in the next section. In chapter 4 we will explore a way that could with time lead to hardware evaluation of arbitrary interaction nets.

3.2 The Interaction Combinators

In [Laf97], Yves Lafont describes an extremely simple interaction system with three types of agents : γ, δ and ϵ. He also proves that this system is universal because any other interaction system can be translated[5] into it. By analogy to the S and K combinators that form a universal subset of the λ-calculus, Lafont named his interaction system the interaction combinators.

Figure 3.1 shows the six reduction rules for the interaction combinators. The $\gamma - \delta$ rule is particularly important for the universality of the system as it is the only one in which agents are created.

The interaction combinators seem very interesting to me because they are very simple locally, and yet remain universal. Since I am trying to compile each agent as a separate entity on a chip, it is vital that each entity have a behavior as simple as possible.

The γ and δ agents are almost identical, and can in fact be interchanged with only slight modifications to the structure of the interaction net. The difference is however necessary for translation of arbitrary interaction nets into interaction combinators to be possible. Traditionally[6] gamma and delta agents are given very different functions.

A γ agent is used as a multiplexer, to group information in bundles. Its use closely resembles the use of $\parr$ and $\otimes$ in chapter 2.

[5] A translation is a transformation that replaces each agent of an interaction net by another interaction net.

[6] The use of γ and δ that was made in [Laf97] started this tradition.

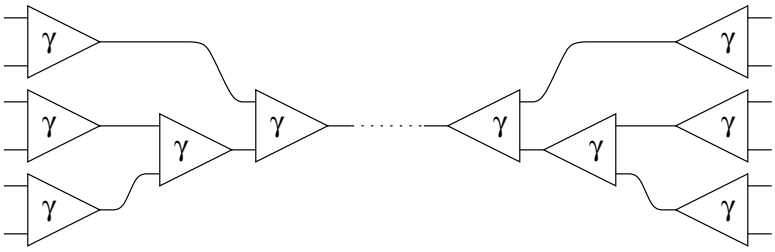


Figure 3.2: With a tree of γ agents it is possible to multiplex together as many wires as one desires. There are many ways to multiplex a given number of wires together, but the only way to demultiplex them is with a second tree that is symmetrical to the first one. This figure shows some trees that could be used for abstraction and application in the encoding of the lambda calculus into interaction combinators.

A δ agent is used to make copies of subnets made up of γ and ϵ agents. It turns out that copies can be made of arbitrary nets by pulling the δ agents out of the net before copying. The encoding of the λ-calculus will give an idea of how this can be done.

In the sequel, it will often be useful to multiplex more than just two wires together. This can be done easily with trees of γ agents (see figure 3.2). Care has to be taken because in general the multiplexer is not identical to its corresponding demultiplexer. Instead the auxiliary ports of the γ agents have to be permuted. Any tree of γ agents with the right number of ports can be chosen, but balanced trees are likely to reduce faster in parallel implementations.

3.3 The Geometry of Interaction Interpretation

In the present chapter, I will be trying to compile interaction combinators into hardware using geometry of interaction ideas, once again. In this case we have an algebra that is just slightly more complicated than for the multiplicative fragment of linear logic.

Indeed, we now have two types of multiplexers, one of which is twisted. We will give each type of multiplexer two algebra elements that we will call : p_γ, q_γ, p_δ, q_δ. The δ rules will behave just like the $\otimes - \parr$ rules :

$$p_\delta^* p_\delta = 1,\ q_\delta^* q_\delta = 1,\ p_\delta^* q_\delta = 0,\ q_\delta^* p_\delta = 0$$

For the gamma rules, the twist in the rule has disappeared which makes for a slight change :

$$p_\gamma^* p_\gamma = 0,\ q_\gamma^* q_\gamma = 0,\ p_\gamma^* q_\gamma = 1,\ q_\gamma^* p_\gamma = 1$$

Since the $\gamma - \delta$ rule makes γ and δ agents move through each other without interfering, γ subscripted elements commute with δ subscripted elements. Because of this commutation rule, when we have a product of algebra terms, we can move the γ elements to the front of the product, and the δ elements to the

Figure 3.3: On the left is a net that is being built. It still has free variables, and it has a port through which it makes copies. On the right is a net that has been finalized by putting a copy operator (a δ agent) on its copy port. Note that the finalized term has no variables. Indeed, the terms that would be plugged into the free variable ports must be δ-free and there is no room left to connect the variable's copy port. It would be possible to leave room for each free variable to have a copy port. For simplicity, and without loss of expressive power, I will only consider finalized terms without any free variables.

back. This decomposes the algebra into two parts that can each be modeled by a stack, as in the logic net case. All the details can be found in [Laf97].

In this way the geometry of interaction has shown us how to evaluate a net of interaction combinators using a token that carries two stacks. The rest of this chapter will try to put this into hardware.

3.4 Encoding the Full λ-calculus

To encode the λ-calculus with interaction combinators, I used an encoding from [MP01]. The encoding I used is the so called *call-by-value* or $!(A \to B)$ encoding.

In chapter 2 the abstractions and applications of the linear λ-calculus were represented by the $\otimes$ and $\parr$ of linear logic, acting as a multiplexer-demultiplexer pair. Now we wish to move on to the full λ-calculus. For this we need to be able to copy and to erase information. As was seen in section 2.8, erasing isn't a big problem. With the interaction combinators, we just have to add ϵ agents on branches that we wish to erase.

Copying is a much trickier problem. δ agents can be used to copy nets that are only made up of γ and ϵ agents. But this doesn't tell us how to copy a net that contains a δ. The trick to solving this problem, is to leave the copying agent out of the net. This is done by adding copy ports to the interface of the net. The net is built with the assumption that when it is used, a copy agent will be connected to these ports. The net in itself can in this way be made up entirely of γ and ϵ agents, and therefore be copyable, and still have the ability to make copies. The encoding of the λ-calculus will make this much clearer.

Because of the copy-extraction that I have just described, compiling a lambda term will be done in two steps. In the first step, a δ-free net will be generated. It will have a principal door at the top, auxiliary doors at the bottom for free variables, and copy door on the right hand side. Once the whole lambda term is built, it can be finalized into a working interaction net, as in figure 3.3, by adding a δ agent on the copy doors.

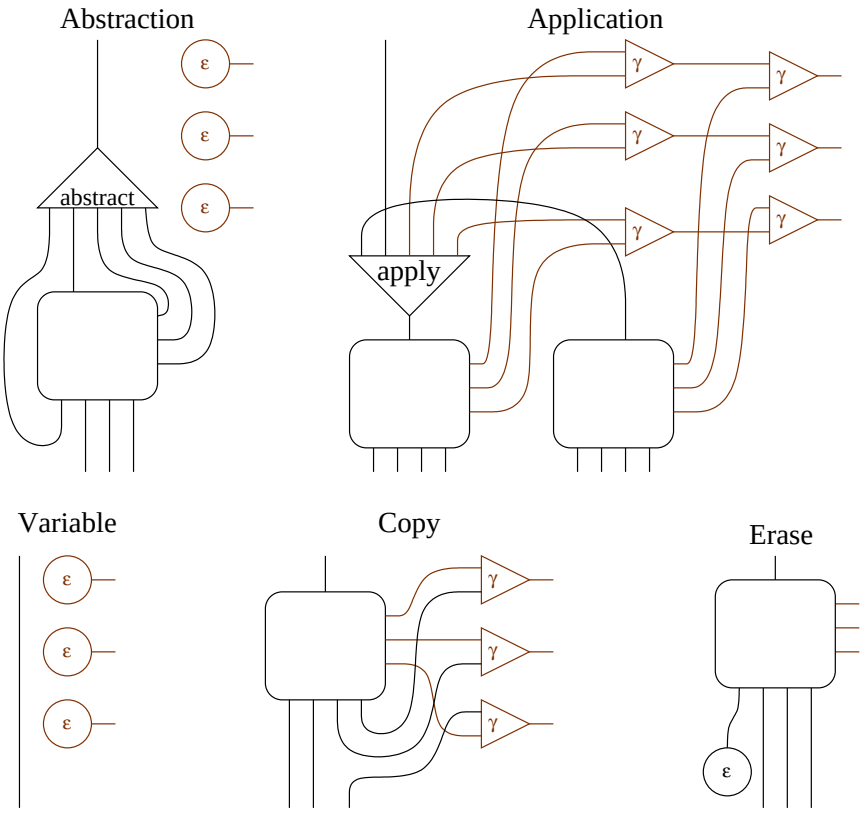


Figure 3.4: Encoding a lambda expression with interaction combinators is done using a few elementary structures.

I will now explain how to compile each part of a λ-term. The reader can follow the different steps in figure 3.4.

To encode a variable, we directly connect the principal door to the auxiliary door. Since no copying is necessary, the copy doors are connected to ϵ agents.

The coding of abstraction is constrained by the fact that we would like to be able to copy a term simply by making a copy of its principal door. This means that we have to make the copy door of the unabstracted term come through the principal port of the abstracted term. We will do this by multiplexing together the abstracted term's principal door, the abstracted variable's door as well as the copy doors. We then have to add epsilon agents to fill the unused copy doors of the new net.

To encode an application, we demultiplex the principal door of the function. As in the linear case, we connect what corresponds to the variable to the parameter term, and what corresponds to the the principal door becomes the principal door of the new term. Unfortunately we now have two copies to make, one from the parameter term, and one that was demultiplexed from the function term. To make both copies with only one set of copy doors, we simply multiplex the sources together as well as the respective destinations. We can now make both

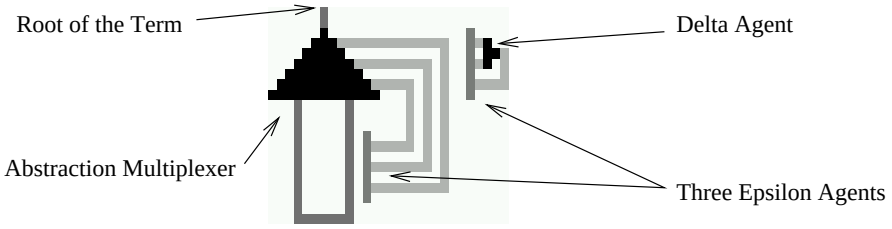


Figure 3.5: The identity, $\lambda x.x$, compiled into interaction combinators. We can see an abstraction, a δ agent and two groups of three ϵ agents.

copies with just one set of copy doors[7].

The encoding described above is sufficient to encode a linear lambda term. To encode any term we still have a little work to do. When encoding abstraction, I implicitly considered that exactly one auxiliary door was being abstracted. What if the variable that we are abstracting appears more than once, or not at all? Variables appearing more than once can be merged together by using a copier. If many copiers are necessary, they can be combined with multiplexers in a similar way to the two copies that appear in an application. In the case of unused variables, an ϵ agent can absorb the incoming variable.

3.5 Simplified graphical representation

After spending a lot of time hand-drawing compiled lambda terms, I decided that an automated drawing tool would be useful. This section describes the graphical conventions that were used by the one that I wrote, and that will be used for the figures in the following sections[8].

I designed my drawing routines to draw agents very small in order to allow easy visualization of large lambda expressions. The figures included here have all been enlarged. Since patterns of γ agents such as abstraction, application and sharing copy doors occur very often, they have been represented in a compact manner. A few examples should get the reader familiar with the representation.

Figure 3.5 shows $\lambda x.x$ compiled into interaction combinators. The abstraction multiplexer is represented as a triangle pointing up. The variable and the result enter through the bottom of the triangle, while the copies enter by the right hand side. This is done to make the structure of the lambda term stand out. Also note that for the same reason, the copy wires are lighter than the main stream wires. The delta agent is a little triangle pointing right. It always appears in exactly the same place so it doesn't have to stand out. ϵ agents are shown as a darker segment blocking off the end of a wire, in many cases they are run together to save space.

In Figure 3.6 we have a slightly more complex term. There is an application, a triangle pointing down, with the same disposition as the abstraction. There are also a couple of triple γ agents, that are used to share copies. The three gammas are represented as one triangle pointing south-west. Wires coming from

[7]What we have done here is simply copy a pair instead of copying the sides of the pair separately.

[8]The figures with large pixels are computer-generated.

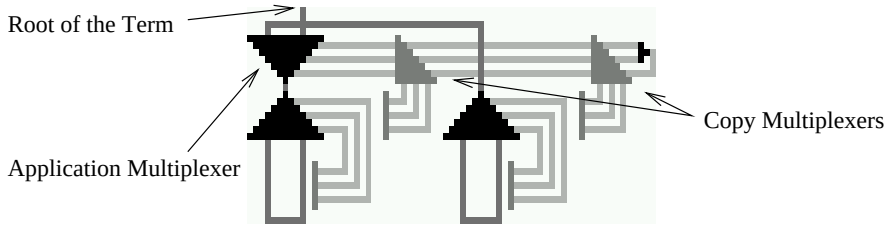


Figure 3.6: The identity applied to itself, $(\lambda x.x)\ (\lambda x.x)$, compiled into interaction combinators. We can see an application, and some triple gamma agents used to share the copy door of a term, as well as the agents that we had already met up with.

the left are paired with wires coming from the bottom by a γ agent into wires emerging from the right. The top, middle and bottom wires match with the left, middle and right wires, respectively.

3.6 Adding the constants

Constants are added much as they are in the linear case, since the constants can simply be seen as extra agents. Base values are an agent with no auxiliary ports, and constant functions have one port per parameter and one for the result. Initially, a constant function has its principal port directed towards one of its parameters. When the parameter arrives, the constant function is transformed so that it is ready to accept a new parameter. Lafont gives some specific encodings in [Laf95], others encodings, which use infinite classes of agents, can easily be developed, and are better suited to real life applications[9]. The main difference is that only the gamma stack is empty when a value of base type is reached, as the delta stack must retain information about which of many paths led to the value. This means that there is no need for rules of interaction between γ and a constant. Rules will however be necessary for encounters between δ or ϵ agents and constants. These rules will simply copy or erase the agent as in figure 3.7.

There is however an extra constant to be added, which is the *fixed-point combinator*. The fixed point combinator can actually be encoded with γ and δ agents. No specific agent is necessary. Figure 3.8 shows the encoding of the fixed point combinator. We could actually do without the fixed point combinator since it can be encoded directly in the full lambda calculus. However, such encodings are very inefficient.

3.7 What I Worked on

As far as the interaction combinators are concerned, I tried out a variety of ideas. None of them were particularly successful so I finally moved on to the

[9] An integer, for example would be represented by zero instead of being represented by the n^{th} successor of 0

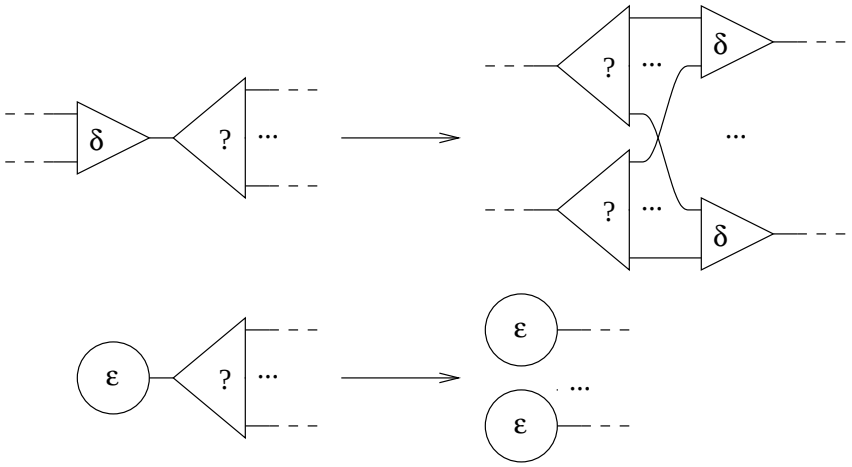


Figure 3.7: A δ agent copies whatever it meets except another δ-agent. An ϵ agent erases whatever it meets.

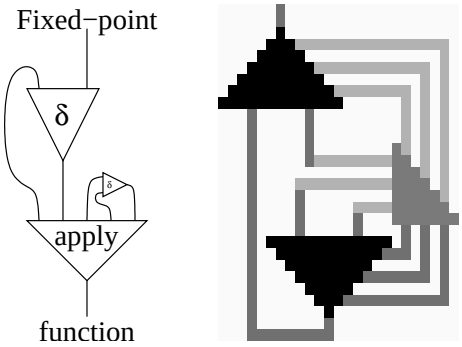


Figure 3.8: On the left, is an encoding of the fixed-point combinator that shows what we are trying to do : copy the result of an application and use it as a parameter to the same application. On the right, is the same net with the copy extracted, and with the function that is to be applied extracted, as it is generated by my program.

interaction processor concept. Some of these results could be of interest to others who might explore computation with interaction combinators.

3.7.1 Direct Stack Based Implementation

In order to observe term evaluation, and to try out some optimizations that will be covered in the next section, I wrote some code to compile a λ expression into Esterel using two global stacks. This implementation also used a global register to store integer and boolean constants, and had only one incrementer and decrementer that operated on them. Though it is quite slow, and impossible to parallelize directly, this implementation shows a sequential encoding of a geometry of interaction into hardware that could easily be generalized to most geometry of interactions. It is the hardware equivalent of Mackie's *geometry of interaction machine*, from [Mac95].

The resulting circuit is in two barely connected parts. On one hand the data part keeps track of the stacks, and the constant values. On the other hand, in the sequencing part, a token flows through a circuit whose topology is very close to that of the interaction net. As the token moves around the sequencing part, orders are sent to the data part to push or pop a value, or to do an operation on the value register. When it reaches a fork, the data part tells the sequencing part which way to proceed.

In a geometry of interaction machine[10], there is a difference in the sequencing part, since it is executing assembler code rather than having a token flow around. The data part, however could be identical for both machines.

I find this resemblance quite inspiring though I do not really know what to do with it.

3.7.2 A Trivial Optimization of the Encoding

Before producing an electronic circuit, it is profitable to do as much optimization as possible. Let us look at two simple rules that can drastically reduce the number of agents that are needed for a circuit.

One sub-expression that frequently turns up in expressions is $(\lambda x. \ldots)(\ldots)$. In the λ-calculus terminology, this is called a redex. With interaction combinators, it is encoded as a multiplexer running straight into the corresponding demultiplexer. It would be foolish to leave such a structure in an circuit, since nothing is lost by doing the reduction at compile time[11]. With a little generalization we find that much useless work can be avoided by doing all the $\gamma - \gamma$ reductions at compile time. This reduction phase terminates, just as in the linear case, because each reduction reduces the number of agents by two.

In fact, further optimizations can be done by normalizing the net while excluding $\gamma - \delta$ reduction, though in fact the $\gamma - \gamma$ rule is the one that gives the most improvement[12]. With these reductions, we are getting rid of some structure that was imposed by the λ-calculus, but that didn't play an important

[10] It must not have been optimized to jump straight back after finding an answer

[11] Indeed, in the one-hot encoding of chapter 2 we were effectively doing all the multiplexer reductions at compile time, which could at times give an exponential reduction in geometry of interaction path length.

[12] In fact, the $\delta - \delta$ reduction will never be applicable.

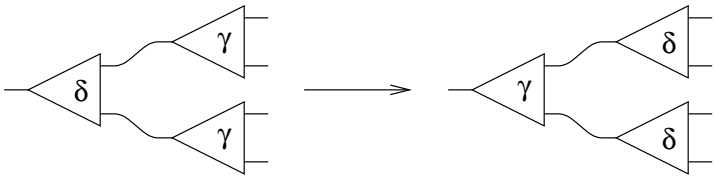


Figure 3.9: By exchanging γ and δ more reduction can be done at compile time, leading to greatly optimized circuits.

part in the computation. We have rearranged the pipes to make them shorter, but the appliances haven't changed in any way.

Further optimizations can be obtained by doing some non-interaction-net reductions. Two γ agents back to back can be reduced to a wire and two δ agents back to back with a twist in the connection can likewise be reduced. We can also replace a γ or δ agent with both auxiliary ports connected to ϵ agents by an ϵ agent. These reductions are easily seen to be valid with the geometry of interaction.

Still, it turns out that as terms are evaluated, much time is spent multiplexing wires together to go through the single delta operator, just to be demultiplexed again. The presence of the delta prevents the reductions that we have discussed above. In fact, if we draw the interaction net as a set of trees of agents with principal ports pointing down, we find that after the optimizations described above, we have exactly two trees. They are entirely made up of gamma agents, except for one delta agent at the root of one of the trees, preventing any interaction from happening[13].

It is clear that we cannot start using the $\gamma - \delta$ rule[14] because it would cause the net to grow, possibly without bound. There is, however, a change that leaves the size of the net unmodified and that unblocks the situation. The idea is to replace a δ agent with γ agents connected to its auxiliary ports by the same configuration with γ and δ exchanged, as in figure 3.9.

After all the above reductions have been exhausted, we end up with an interaction net where all active pairs are $\gamma - \delta$ pairs, and each δ has at least one auxiliary port that is not connected to a γ principal port.

3.7.3 Typing Interaction Combinators

Since typing was so useful for compiling the linear λ-calculus, I spent some time exploring type systems for the interaction combinators. Two type systems I explored are shown in figures 3.10 and 3.11. I refer to them by γ and $\gamma\delta$ respectively. It is easy to make interaction nets that are untypable with both of these systems by connecting a gamma agent's principal port to one of its auxiliary ports. Typing also fails on lambda terms. With the $\gamma\delta$ type system,

[13] Indeed, initially, there was only one δ agent, and no new δ agents were created by the previous reductions. The only exception is for terms that don't actually make copies, where the δ agent gets removed by ϵ agents. Since the only reduction that hasn't been done is $\gamma - \delta$, if there is anything left to do it will be such a reduction. And there can only be one active pair since there is only one δ agent.

[14] At least not as a simple optimization strategy

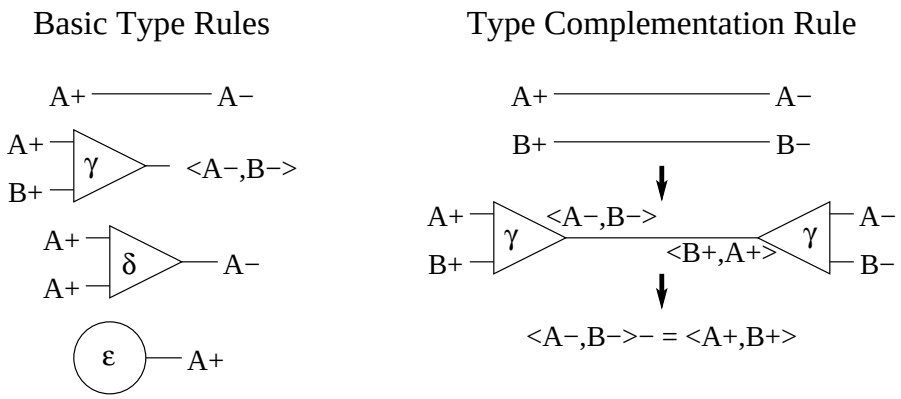


Figure 3.10: A type system that is almost suited to λ-expressions compiled into interaction combinators. A delta agent can only make copies of identical type. On the left are the typing rules for the different agents. On the right is an extra rule that is used to complement a type, and that is deduced from the $\gamma - \gamma$ interaction rule.

any recursive λ-term fails to type. With the γ system, describing which terms is more subtle[15], I would not have found any without writing the compiler first.

A direction that I have good hopes for, but haven't had time to explore, is to use the $\gamma\delta$ type system, allowing a few wires with typing errors. The whole term can then be compiled with the one hot encoding, with re-entrant casters (which will need two stacks, most likely) to handle the type mismatches. Perhaps more imaginative type systems can actually make the casters become part of the type system[16].

It is hard to say much more without writing more code and doing more tests, as the types get very large very fast because of the copy doors.

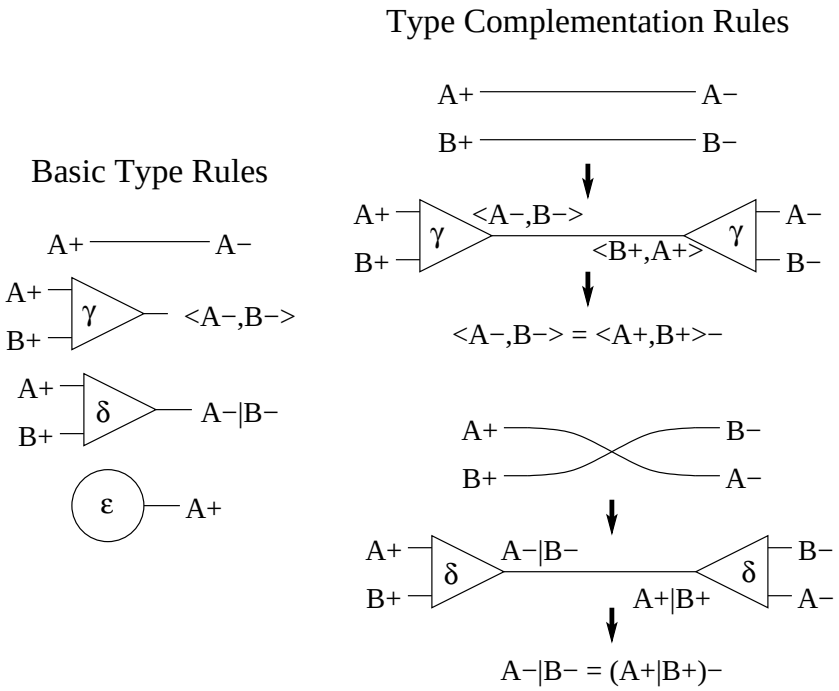
3.7.4 Understanding the Stacks

To try to optimize the simple two stack implementation of the interaction combinators into hardware, I did some exploration of the operations that take place on the stack during computation. To do this exploration I wrote a program to generate a graphical representation of the term compiled into interaction combinators, and then to be able to step through execution while watching the token

[15]I originally expected the γ type system to work for all typable λ-terms, and thought that the typing errors I was getting were a bug because they appeared in strange ways. Then I realized that since this type system allows compilation into a single stack finite automaton, the terms that it types cannot be Turing complete.

The actual typing error occurs when the δ agent is added, so there should be a way to get around it with a little work (including an extra stack). The error occurs when a function is applied to a term that contains another copy of the same function. Indeed, in that case, the type of the copy part of the function must be included in itself. So unless the function makes no copies at all this situation will lead to an error.

[16]A type system with a type $\mathcal{A}$ such that $\mathcal{A} = \langle \mathcal{A}, A \rangle = \langle \langle \mathcal{A}, A \rangle, A \rangle$ would allow situations that cause type conflicts to enter the type system. It is not clear, however that the resulting type system would have a natural encoding in hardware, as it would certainly require at least two stacks to function.



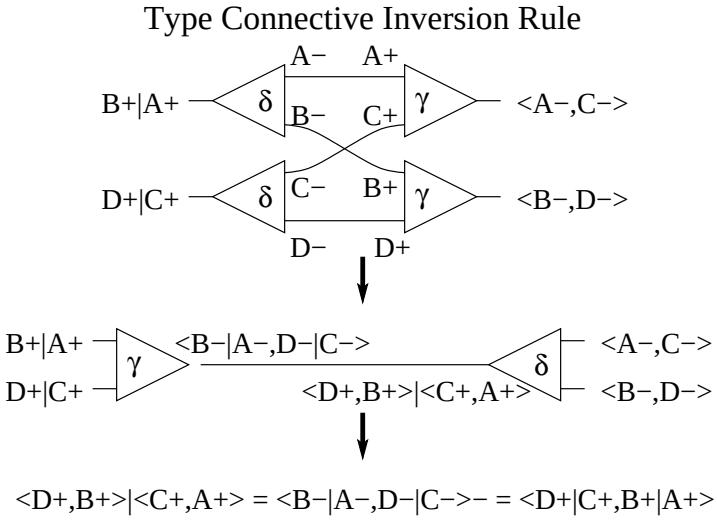


Figure 3.11: A type system that treats γ and δ connectives as similarly as possible. The top left quarter rules show how to type each agent. The top right quarter derives type complementation rules from the $\gamma - \gamma$ and $\delta - \delta$ rules. The bottom half derives a rule that allows the permutation of type connectives.

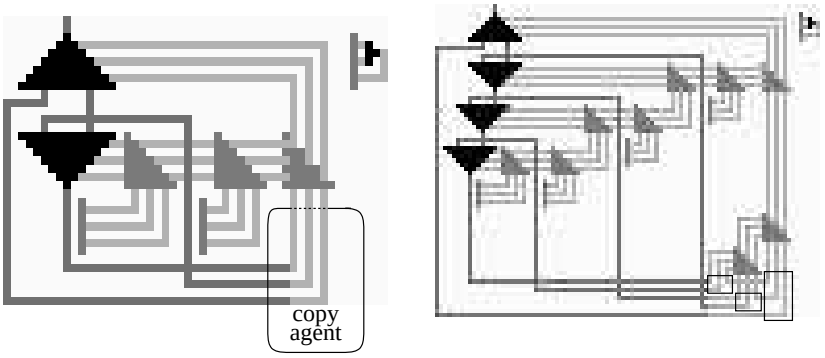


Figure 3.12: On the left, the lambda term $\lambda x.xx$ is represented, with its copy agent explicitly shown. The bottom wire on the left hand side is the variable that is being copied, and the wires above it are the copies. The three wires at the top are the copy ports that the copy agent uses to actually make the copies. Copy mode is entered (possibly recursively) when the token enters the copy agent from the left, and is exited when the token exits to the left. On the right, $\lambda x.xxxx$ is shown. This case is a little more complicated because three binary copy agents are needed. The copy agents are represented by rectangles. In two of the rectangles the source of the copy enters through the left of the box. When a copy of the variable is made in this case, copy mode is entered and exited twice.

flow through the net[17].

Two Modes of operation

The first noticeable fact is that there are two quite distinct modes during computation. In the first mode, that I will call *normal mode*, the gamma stack is used in exactly the same way as the single stack of chapter 2. In particular the copy ports of the abstraction and application agents are never used. In normal mode the δ stack is never changed.

The second mode, that I will call *copy mode*, is entered whenever a variable is used more than once. When the variable is used exactly twice, it is entered on leaving the variable port of an abstraction and is exited just before the use of the variable. Is is also entered and exited in the same places when the token is traveling in the other direction. The transition corresponds to the grayed paths in the graphical representation that my program generates. If we were to draw the interaction net with copy agents drawn instead of uses of the copy door of the term, copy mode would be effective between the moment when the token enters the copy agent and the moment when it leaves it. This can be seen for two or more copies of the variable in figure 3.12.

The goal when passing through copy mode is to add or remove a mark from the δ stack. When going from the variable's use to the variable port of the abstraction, a mark is added to the δ stack to say which use of the variable the token will have to return to. When going in the other direction, the mark is

[17]Interaction nets represented by this program have already been shown in section 3.5.

removed, and the token exits copy mode at exactly the place where it entered.

When in copy mode, the token gets multiplexed a number of times, then enters the delta agent, or a copy port of an abstraction. In the latter case, it flows around the net and ends up leaving a copy port of an application agent, at which point, it will once again be multiplexed before entering a delta or an abstraction copy port. In the end the delta agent is always reached. During all this, the gamma stack from before copy mode was entered is unaffected.

The path that the token follows to get to the delta is very important. Indeed, the mark that is added or removed when passing through copy mode isn't in general placed on the top of the stack. This means that some marks have to be removed before the mark that we are concerned with can be manipulated. These marks then have to be replaced.

These marks that have to be removed then replaced are found when the token goes between a copy port of an abstraction and a copy port of an application. During this time, the token can go through a number of copy operations, as if copy mode were entered recursively, it is with such operations that tokens are removed from the delta stack and then replaced.

As the token goes towards the δ agent, it crosses a number of multiplexers (south-west pointing triangles in my graphical representation). These multiplexers all add information to the γ stack that is used on the way back from the copier. In particular, when the fixed-point combinator is used there are cases in which a number of δ-marks that depends on the depth of recursion has to be removed, the gamma stack, in counterpart, is filled with marks that allow the token to return to the same level of recursion after the copy is made.

To sum up the use of the stacks, during normal mode, the δ stack is untouched, and the γ stack is used as in the linear case of chapter 2. In copy mode, which can be entered recursively, the δ stack is emptied to a certain extent, a change is made, then it is filled up again, while the γ stack records the path that was followed in order to be able to retrace it. Since knowing the path that is followed while going towards the δ agent is sufficient to be able to reconstruct the marks that were removed from the γ stack, it is as if the information from the δ stack was being recorded on the γ stack. On examples such as

$$Y(\lambda xy.\text{ifte } y(xy)y) \text{ true}$$

that could be written in Caml as

```
let rec loopiftrue c =
  if c then
    loopiftrue c
  else
    c
in loopiftrue true;;
```

this flow of information, back and forth, between δ and γ stacks is perfectly visible. Especially as recursion depth gets deeper[18].

[18]To find an example like this, all that is needed is a non-terminating recursive function that uses its argument.

The Path to the Top

The above discussion suggests that much time could be gained if less time was spent removing elements from the δ stack just to put them back on after a little change. This is particularly true since when an deep element of the delta stack is accessed, the first element to be removed above it isn't necessarily the top of stack. In many cases another deep element is removed first. This means that the first deep element is unearthed, it is removed, the elements that were above it are placed back on the stack. Then it is time to remove another element that is above the one we want to deal with. Of course all this is happening recursively so getting to a mark buried in the δ stack can take a very long time.

It is my belief that a better data structure than a stack would be useful for the information in the δ stack. Structures such as the tree used by the Geometry of Interaction Machine of [Mac95] are good candidates since they are known to efficiently store similar types of information. Unfortunately I was unable to make sufficient progress in understanding the path that is followed during a copy operation to make this into something concrete. And I do not know if the resulting system would be easy to implement in hardware. In any case, I do not expect much parallelism to come from this direction since it still relies on a centralized, unbounded data structure.

If I was to further pursue this search, I would look at the relation between the path that is followed during a copy, and the element of the algebra of the full geometry of interaction that the token is currently at.

An interesting result of this exploration is the graphical interface to display lambda terms compiled into interaction combinators. It is more fully described in appendix C.

Chapter 4

The Interaction Processor

4.1 The Concept

The ideas of the previous chapters have relied on token passing and the geometry of interaction to try to compile lambda expressions into hardware. In each case however, I have been unable to make room for much parallelism. In chapter 2, binary operators such as addition could be made to lead to a limited form of parallelism. In chapter 3, I was unable to even allow such a limited form of parallelism to appear because of the use of global stacks. At best, the parallelism that I could hope for was that coming from binary operators. There was always a great deal of sequentiality imposed by the token passing. Much of the parallelism that that exists in interaction nets, where reductions can occur in many parts of the net simultaneously, was quite inaccessible.

To try to solve this problem, it would be nice to be able to actually do graph reduction on an interaction net. This is quite difficult because every given agent can potentially end up connected to any other agent, which means that each agent must be wired to be able to communicate with every other agent[1]. Because of this, it seems that graph reduction requires that each pair of agents be able to exchange messages, which requires a very large network.

Because a network that interconnects the agents is likely to take up a significant amount of the chip, and to take a considerable amount of time to compile, it appears that it would be useful to have a generic routing block, that is used for each compilation, and a small custom part that determines the initial connections between the agents.

By going just one step further, the custom part can be done away with all together, so that we have a chip with an array of generic agents, and a network that interconnects them (see figure 4.1). To evaluate a particular net, all that has to be done is to load the connections of that net into the generic component, and then let it reduce by having connected agents exchange messages.

What I have just described is a sort of processor for interactions that I have decided to call the *interaction processor*. For the rest of this chapter, I will give

[1]It might actually be possible to consider a way for agents to be moved around the circuit, thus limiting the need for communication. However I don't know how to do it that way, and I suspect that one would end up with a general purpose machine like the one I am presenting here anyhow.

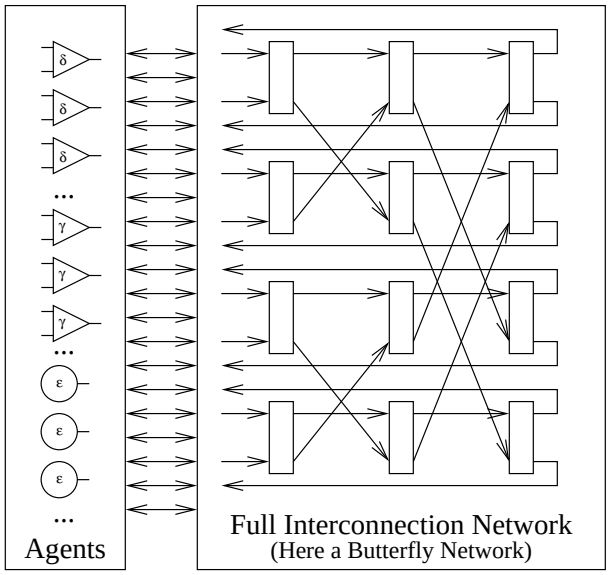


Figure 4.1: The *interaction processor* is made up of a number of configurable agents, interconnected by a network that is able to pass messages between any arbitrary pair of agents. A given interaction net is loaded into the processor by telling agents who they are connected to. The interaction net is then reduced by having connected agents exchange messages. The result can then be extracted from the reduced net.

more details on how an interaction processor could work, in particular in the linear case for which I have a working simulation.

4.2 The Gamma Processor

There are a number of challenges to meet to get a fully functional interaction processor. First of all, connected agents have to be able to detect if they are connected through their principal ports, and if they are, they have to be able to interact. In the general case, interacting can mean creating new agents. This calls for some scheme to allocate unused cells. The allocation has to take place in parallel or else reduction will become sequential. If there is allocation, then there has to be erasing, or else the free agents will be rapidly depleted in any practical application. This means that an agent has to be able to detect when it is unused, in order to add itself to the free memory pool.

In this section, we will cover the first aspect, finding active pairs and reducing them, by building a processor that can handle the linear lambda calculus compiled as a net of γ interaction combinators.

4.2.1 Message-passing Infrastructure

In the rest of this section, we will consider that each port of each agent has a unique address. This address is made up of two parts, the *physical address* that is used to identify the agent, and a *sub-address* that is used to identify the port. The type of port (principal or auxiliary) can be directly determined from the sub-address.

The interconnection network will be considered to have the following functionality : each agent (physical address) has an input and an output register. When the output register is empty, the agent can write a message to it, made up of a full destination address, and a number of data bits that is limited for a given network. The network is responsible for getting the whole message to the input register of the destination address. Messages sent first will not necessarily arrive first, we just know that they will arrive eventually.

In order to avoid deadlock, we will require that the agents read incoming messages as soon as they are ready, even if the output buffer for that agent is full.

4.2.2 The naive γ agent

The gamma agent is one of the simplest agents we can get since a reduction is simply the destruction of two agents. No agent creation is necessary. Nevertheless we will see that this simple case isn't as trivial as one could imagine. In the following paragraphs I will describe a system that almost works, but that has a race condition, that we will then see how to solve.

Initially each gamma agent knows who it is connected to. As interactions occur, some connections change. A gamma agent whose connection changes will be sent a message by the agent that is responsible for the change, so that each agent can keep up to date information on who it is connected to.

Since the type of a port (principal or auxiliary) can be directly determined from its address, an agent knows immediately when its principal port is con-

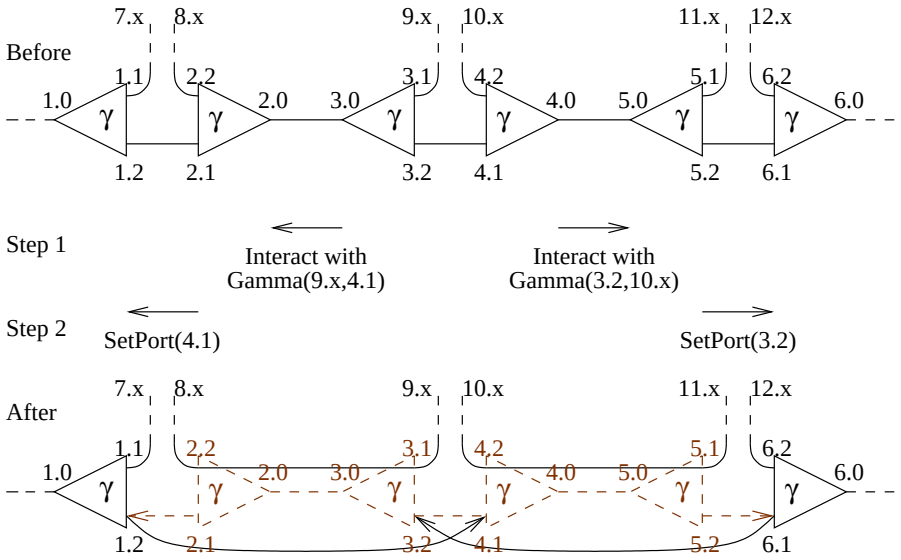


Figure 4.2: In this figure, we follow some of the message passing that leads to the erroneous reduction of the interaction net shown at the top. The resulting net, shown at the bottom is incoherent, as we can see from the arrows that represent incoherent wires. The agents that should have been removed from the net are in dotted grey. In the middle, the messages leading th the error are shown. They are exchanged in two successive steps. In the example, two interactions are taking place at once. In the first one, between 2 and 3, it is found that 1.2 and 4.1 must be connected together. But by the time the connection is made, 4.1 is no longer in use because of the second interaction. The same problem arises in the interaction between 4 and 5.

nected to an other agent's principal port. Thus determining when it is time to interact is very easy.

When it is time to interact, each agent in the interacting pair send a message to its counterpart, telling it what its auxiliary ports are connected to. An agent receiving such a message then knows exactly who it is interacting with, and knows what the net will look like after the reduction. The changes that have to be made to the net are then carried out. Since the interaction is symmetrical (both agents are γ agents, and both have exchanged the same type of messages) the work that has to be done to change the net has to be split between the two agents in a symmetrical way. For example, an agent can be responsible for telling whoever is connected to its auxiliary ports that they are now connected to the corresponding auxiliary ports of the other agent. In this way, γ reduction takes place, and the two interacting ports are removed from the net.

The message passing strategy above is simple and gives a good idea of what we want to do in the gamma processor. However, it doesn't work when things are happening in parallel. Indeed, the interactions do not take place in an atomic way, but through message passing. Difficulties occur when two agents connected by auxiliary ports interact simultaneously, because of a race condition. Figure 4.2 demonstrates the problem.

The problem arises because when two auxiliary ports are connected together, both sides can interact simultaneously, which makes keeping the net coherent difficult. This is the only difficult case. When a principal port is connected to an auxiliary port, the principal port is sure not to change, so the auxiliary port can change safely. When two principal ports are connected, they will reduce hand in hand so that the net remains coherent.

If we think back to logic nets, we realize that it is in fact exactly the axiom links that are causing us problems. One idea would be to add axiom agents to the net[2], that would keep two conflicting interactions separated. Axiom links would have to disappear when connected on one side to a principal port, and two or more axiom links connected one after the other should reduce into a single axiom link. In cases where many axiom links get connected one after another, a good reduction scheme should reduce the whole chain in $O(\log n)$ time. I was however unable to find a good reduction scheme that doesn't flood the interconnection network or require unlimited buffers in the agents, which is clearly unacceptable.

4.2.3 A correct γ agent

For the implementation I wrote, I settled for a slow reduction scheme that avoids the explicit use of axiom agents. There is a slight hitch, however, because reduction only takes place to let a principal port through. Essentially, the part that was to be played by the axioms is played by active pairs of γ agents instead.

The reduction method that I am about to describe relies on the following assumpiton : *When a port is connected to a principal port, it will remain connected to that principal port except if its own agent decides to make a change.* This is a reasonable requirement because changes only occur in the net when a principal port is connected to a principal port. Therefore an auxiliary port can get changed at any time because of the principal port that it is connected to, while a principal port only changes by interacting. The port that a principal port is connected to can thus be sure that any change to the connection will come from itself.

The counterpart of this assumption is that it is possible to send the address of a principal port in a message. Conversely, we saw in the naive algorithm that sending the address of an auxiliary port could lead to trouble because the auxiliary port can change while the message is being sent. Sending a message to an auxiliary port is required to make interaction possible, but the sender has to be able to deal with the case where the connection is changed before the message arrives.

But how can we do $\gamma - \gamma$ interaction while only sending the address of a principal port? The trick is to wait for a principal port to arrive at one of the auxiliary ports of the γ pair, and to forward the address of that principal port through the γ pair at that time. Figure 4.3 illustrates the scheme. So in fact, the γ pair is acting like two wires, that only let principal ports through.

Going through a barrier of γ agent pairs takes an amount of time that is linear with the number of pairs, instead of logarithmic, which would be the best possible. If principal ports are present at each end of the chain, they meet in the middle.

[2]If the axiom links are symmetrical we would no longer strictly have an interaction net.

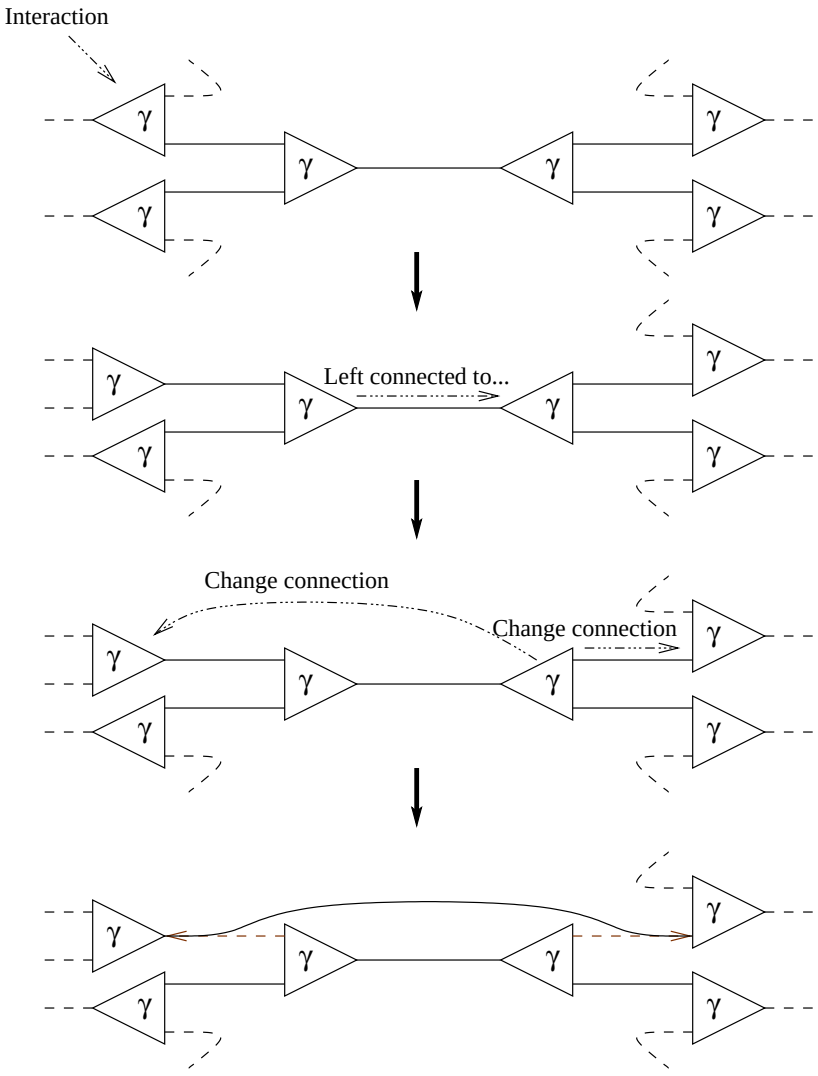


Figure 4.3: Here we can see a simple $\gamma - \gamma$ interaction. As long as the active pair is only connected to auxiliary ports, nothing happens. As soon as a neighboring interaction takes place and a princpial port arrives on the interface, the principal port is connected through the $\gamma - \gamma$ pair. The pair then returns to inactivity until another principal port appears on its interface.

An interesting improvement would be to find some way for the gamma chain to reduce as much as possible before a principal port arrives, if possible in logarithmic time.

4.2.4 Integers and Adders

After writing a simulator for the correct γ processor, I decided to add adders and integers in order to make testing easier. Indeed it is simpler to test nets that reduce to a value of base type, particularly if nets only reduce when a principal port is present.

In my implementation I added four types of agents. A special reader agent, with no auxiliary ports, that is placed on the interface of the net where we expect to get an integer result, an integer agent, an adder agent, and an agent that add a constant to an integer. In hardware, the adder agent is implemented in the same type of cell as the agents that add a constant.

For properly constructed nets only a few types of interactions are possible. The new agents should never have to interact with a γ agent[3]. The reader agent only reacts with an integer agent[4]. An adder agent has its principal port connected to one argument, and two auxiliary ports for the other argument and the result. When its principal port meets an integer, it transforms into an agent that adds a constant. This agent has its principal port on the other argument, and its single auxiliary port connected to the result. When its result interacts with an integer, the integer is changed to reflect the apropriate sum, and then the integer is connected where the result goes (only once a principal port is connected to the result port). Figure 4.4 shows the message passing in a simple case.

4.3 Interconnection Network Architecture

Because of its sheer size, the interconnection network deserves quite a bit more attention that I was able to give it. In this section I will discuss what I used for my implementation, as well as a few design tradeoffs that I have thought about.

4.3.1 Serial versus Parallel

In my Caml simulation, I implicitly considered that messages were transmitted in parallel. In an actual hardware implementation this could be very costly as a message must be able to carry at the very least its destination address and a parameter address, as well as a few bits that indicate the type of message. If one is willing to increase the message latency, a serial transmission of the message through the interconnection network can be done. This should greatly decrease the size of the network, while slightly increasing the size of the agents because of serial encoding and decoding hardware.

[3] Just as constants never mixed with multiplexers in chapter 2.

[4] When this reaction takes place, it is usually possible to stop the processor, since we have our result.

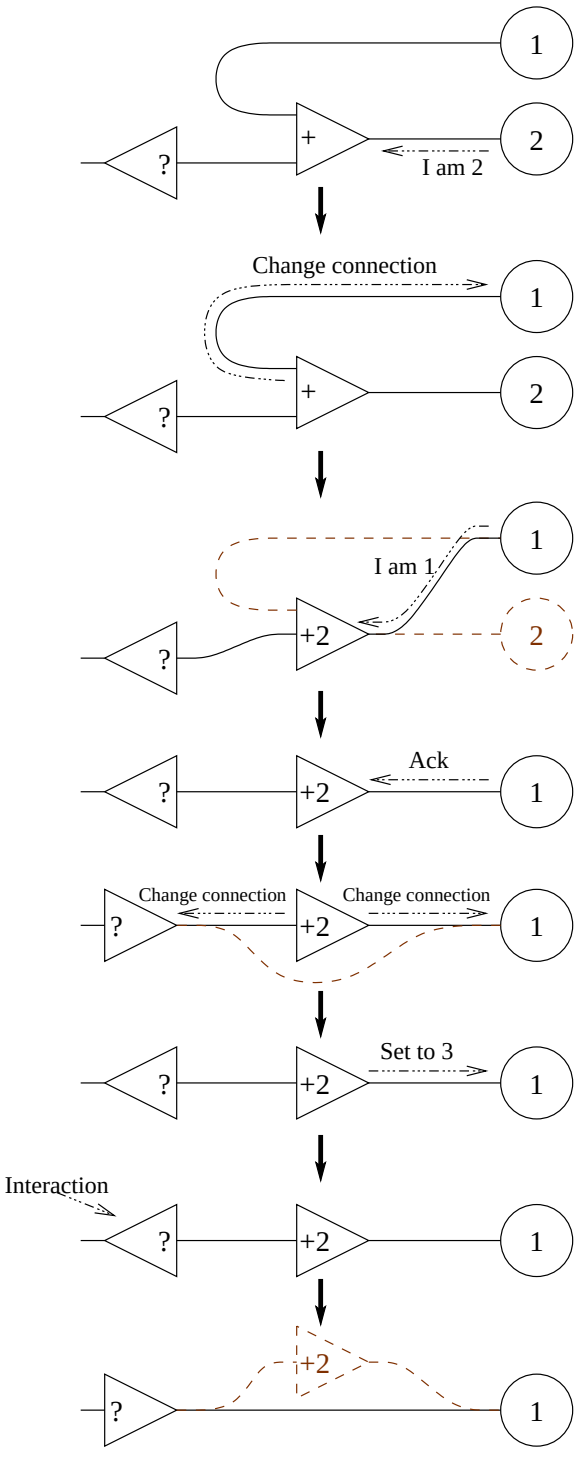


Figure 4.4: The sequence of messages that is necessary to add two integers.

4.3.2 Network Topology

The topology of the interconnection network is a key element that has to be explored and tuned in order to be able to fit a large enough number of agents onto a chip while still having a low message latency.

Butterfly Network

In my simulations I used a *butterfly network* to do the message passing. For n agents, a butterfly network is made up of $\log_2(n)$ layers of $n/2$ binary switches[5]. Each layer corresponds to a bit in the address. Messages at a given layer are routed according to the value of that layer's bit. The network in figure 4.1 is actually a butterfly network for 8 agents.

The butterfly network has $n\log_2(n)/2$ switches, but the best implementations require a surface area of $O(n^2)$[6]. Unless the interaction net that is being reduced is massively parallel, the butterfly network is likely to be used way under its saturation levels. Remember that most agents will only exchange 3 or 4 messages in their lifetime. Thus with as few as 8 or 16 agents, all the messages that are likely to be sent during the whole evaluation can fit in the network simultaneously.

Tree Network

At the other extreme, we could try to use a *tree network* to interconnect the agents. Agents would be at the leaves of a balanced binary tree. A message would be passed up towards the root of the tree until it is on the same branch as its destination. Then it just has to be passed down to the right leaf.

With this approach we have a new problem, though. The way agents are placed in the network becomes very important. Related agents should be placed to minimize activity at the root of the tree. A worst case scenario would be to consider an interaction net in which each message has to go through the root of the tree. In this case computation becomes sequential. Of course the space requirements are much better as the surface area can now be reduced to $O(n\log_2(n))$.

A happy medium could be to combine the two approaches that I have described, by having a butterfly network connect small trees of agents as in figure 4.5. In any case, the smaller the network, the more the placement strategy for the agents will impact on performance.

Omega Network

An interesting idea that I got from [YL] could be applied to the interaction processor. There is a slight variant of the butterfly network called the *omega network*. Essentially, the difference lies in the ordering of the intermediate switches that all of the layers of the network to be identical as can be seen in figure 4.6. We can use the fact that the layers of the network are all identical to do a drastic reduction of our circuit. Indeed, just as was the case with the butterfly network, the omega network is under-used by the gamma processor.

[5]Each binary switch has two inputs and two outputs, and can send an incoming message to each output or delay a message if the desired output is not ready.

[6]See [ACE+96] for details.

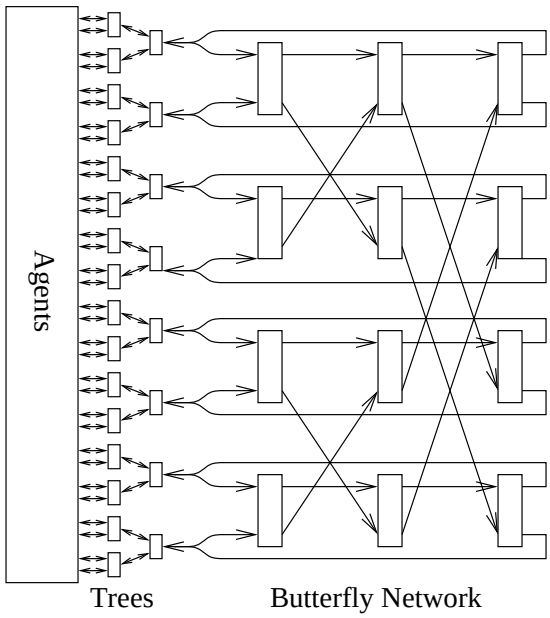


Figure 4.5: With a combination of the tree network and the butterfly a big improvement can be made in the size of the butterfly network, while still allowing bottleneck-free routing at the network-wide scale.

With the omega network this problem can be solved by only including a partial network on the chip. In fact, we only need one layer of the full network to get messages from any given point to any other point.

Figure 4.7 gives an idea of how one omega network layer can be used for an interconnection network. Unfortunately the omega network requires a surface area of $O(n^2)$ per layer because of the large amount of wiring that is necessary. Thus we can only gain a constant factor over the butterfly approach, even though the number of gates has decreased by a larger factor, depending on how big the queues are made.

4.4 Arbitrary Interaction Nets

4.4.1 Allocation

In order to move from the gamma processor to something that is capable of universal computation, it is necessary to have extra agents and rules that create new agents. To create new agents an interaction processor has to be equipped with some extra hardware that allows an agent to allocate an agent of a specific type. This allocation has to be able to proceede in parallel, of course, or the computation will be forced to become sequential. This hardware is entirely work to be done, but for the sequel, I will consider that an apropriate solution has been found in order to describe the other issues related to a general interaction processor.

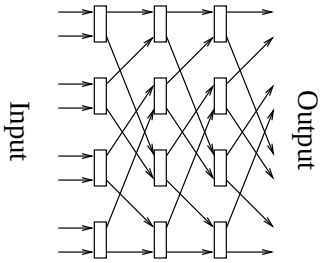


Figure 4.6: With an omega network congestion and routing are very similar to what they would be with a butterfly network. The main difference is that with the omega network, all the layers of the network are identical. Omega networks are rarely used whole though as they take up a surface area of $O(n^3)$.

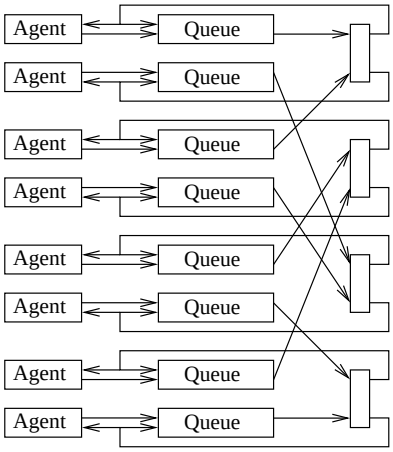


Figure 4.7: With only one omega network layer, it is possible to make an interconnection network. It will only be smaller than a butterfly network by a constant factor, though.

4.4.2 Interaction

For an arbitrary interaction there is a lot more to do that for a $\gamma - \gamma$ interaction. Here is a summary of what has to be done when two agents meet.

1. **First contact** : The interacting agents exchange messages stating the type of the interacting agents. In this way, each agent knows what interaction rule is going to be used. It is a general result in interaction nets that the work of an interaction can be split between the two agents that are interacting. In a heterogenous interaction[7], all the work can be done by one of the agents, for example. In a homogenous interaction, the rule must be symmetrical, so each agent can get a workload identical to the other's. Once the agents know what they are interacting with, we can consider that they each know what work they have to do.

2. **Allocation :** Each agent allocates the agents that it needs to create its half of the right hand side of the interaction rule. It then informs its counterpart of the addresses of the agents that it will need.

3. **Establishing Internal Connections :** All the connections between agents on the right hand side of the rule are established. This must be done with agents in a passive state in order to prevent interaction between misconnected agents. Alternatively, it is possible to connect all the auxiliary ports first, then all the principal ports. Connections through the interface of the rule are made to virtual auxiliary ports of the interacting agents. These virtual ports will forward messages until it is possible to establish a direct connection, when a principal port arrives[8].

4. **Forwarding Principal Ports :** As principal ports appear on the interface of the active pair, they are forwarded to the proper ports on the newly created net.

These steps are summarized in figure 4.8 for a $\gamma - \delta$ interaction.

4.4.3 Erasing

Once we start allocating agents, it is necessary to reclaim agents that are no longer in use in order in order to be able to continue computing with a limited number of agents.

There are cases where determining that an angent is no longer in use is easy. An adder that has finished interacting is certainly free to be reused. For an integer, things are more delicate, since an adder reuses its second parameter to hold the result of the addition. In the case of an integer, the agent that it is interacting with can simply send it a message to kill it if it is no longer going to be needed.

Things are more difficult with γ agents. When an interacting pair of γ agents has forwarded in all four directions, it is sure not to be used any longer. However this usually doesn't happen. Once a principal port has been forwarded through one wire of a gamma agent, it is most likely that forwarding won't

[7] Two different agents are interacting

[8] The virtual ports are necessary because we can't send the address of an auxiliary port through the interconnection network.

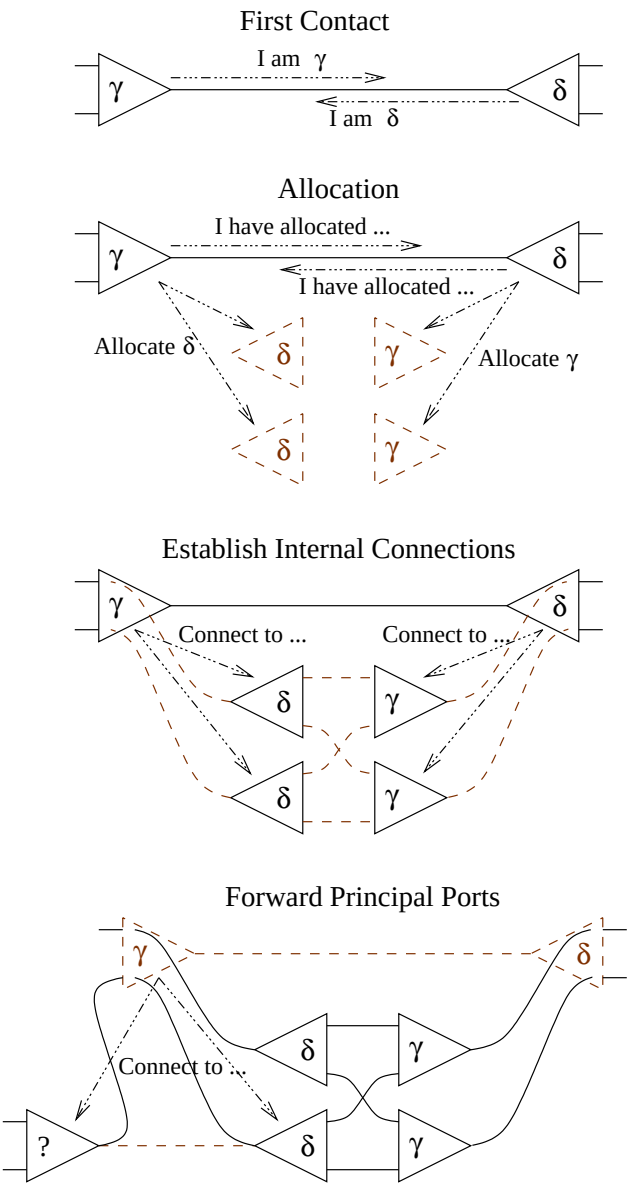


Figure 4.8: The four main steps in a general interaction are illustrated here for the case of $\gamma - \delta$ interaction.

happen in the other direction. The only case where forwarding happens in both directions is when principal ports are being forwarded in both directions at once; forwarding can then take place one step too far before the principal ports get connected directly. In this case, it is possible for the γ agents that do realize that forwarding is done to inform the agents farther along the chain that they won't need to do a particulat forwarding. This strategy can probably be extended for the general case, but once again, this is work to be done.

4.4.4 Reconfigurable Agents

The idea of being able to design an interaction processor for a given interaction system is quite exciting. It might be possible to go one step farther, by designing an interaction processor in which the agents are implemented by a reconfigurable finite state automaton. Such a processor would be able to handle any interaction system whose description fits in the available configuration memory. This idea, is for now entirely in the realm of science fiction, but would be the ultimate step of bringing interaction nets into hardware.

Chapter 5

Conclusion

5.1 My Contribution

Throughout this report, it probably hasn't always been clear what is common knowledge, and what I have contributed. I hope to clarify this in the present section.

The criteria I set up to judge an encoding I worked out for myself. I think they will be of use for whoever might want to work on similar encodings.

In the linear case my contribution was the different ways of implementing the token passing in hardware, as well as exploring a number of exotic but finally unsatisfactory approaches. The *information at token* encodings are directly based on Ian Mackie's explanations to me of the geometry of interaction. I worked out the details of how to implement them, and which encoding is better for what purpose. The idea of using casters is new.

As far as the interaction combinators are concerned, my two stack implementation in hardware is a direct concequence of [Laf97] and [MP01]. I think I am the first to have actually tried out the optimization rules that I describe on λ-terms compiled into interaction combinators. My attempts at typing the interaction combinators to make compiling to hardware easier are also new. I also think that I am the first to have looked in detail at what is happening on the stacks during the evaluation of a λ-term.

Finally, the interaction processor concept is an original development. I hope that the idea will have the chance to be carried further.

5.2 Gained Experience

I think that I have been able to gain much personal experience, on a number of different levels, during my internship.

Being in an active research lab for three months has given me an idea of what research actually is. I have had a chance to do research of my own, and also to observe others. Working from the library was interesting as it allowed me to see much of what was happening in the lab.

The area that I worked in was quite new to me. At the beginning I had an idea of what the λ-calculus was, but Linear Logic, Interaction Nets, and the Geometry of Interaction were all new to me. Because of this, much of my

time was spent documenting myself. I discovered what a valuable source of information scientific papers can be, and what a powerful tool the Internet is to seek them out. For the first time I discovered the importance of bibliographies, that allow a short paper to concisely but precisely build upon previous findings, and that also give a person entering the field access to the key papers that should be read.

Last but not least, I used my internship to explore new tools, some of which I had been planing on discovering for some time. In particular, I moved from *Caml* to *Objective Caml*, learned *Esterel*, started using *RCS* to keep track of my source files and *rsync* to keep my account at the LIX synchronized with my account at home through various levels of firewall. Finally, writing this report was an excellent exercise in *Vim* and *LaTeX*, as well as *xfig*.

Appendix A

Software User Manual

All the code that was written is accessed through a common interface called *lambda2hard*. Commands are read through the standard input and results are displayed on the standard output. Note that there is no prompt, to avoid interfering with output code. The interaction combinator simulator also does some graphic rendering[1].

Commands entered by the user are interpreted line by line. A line can be a λ-expression or a command. Comments are also accepted as in C++ with `//` and `/* ... */`. Text appearing before `/*` will be considered as being on the same line as whatever follows `*/`.

A.1 Entering a λ-term

A lambda term is entered in quite a natural way. For example, $(\lambda xy.yx)(\lambda x.x)$ would be entered :

```
(fn x y -> y x) (fn x -> x)
```

This example shows how to enter abstraction and application, and shows that multiple abstractions can be written in compact notation.

Each time a λ-term is entered, it becomes the current λ-term. Most commands operate on the current λ-term.

Precedence of operators should not cause surprises, but to be sure display the generated term (some extra brackets will be displayed) or enter explicit brackets.

Variable names start with a letter and continue with letters or numbers[2].

Pairs can be formed by using a comma, as in :

```
(fn x -> x), (fn x -> x)
```

[1] If there are problems with the graphics library, the call to the *Termdraw* module can be removed from *parser.mly*, and the program should compile correctly without the *termdraw.ml* and *graphics.cma* files.

[2] When λ-expressions are displayed an initial underscore is prepended to all dependent variables. In order to make re-entering of displayed expressions possible, initial underscores are accepted when entering text. However, using an initial underscore for a free variable is discouraged as it might cause name conflicts in the displayed text.

Church numerals are entered by putting an ampersand in front of the number. $\lambda xy.x(xy)$ can be written `@2`.

A number of constants are predefined :

- Integers are entered as a number.
- Booleans are entered as *true* or *false*.
- One can use *inc*, *dec* and *add* to increment, decrement and add, respectively.
- To test if an integer is zero, use *iszero*.
- The conditional operation is *ifte*.
- The fixed-point combinator is *Y*.
- In some cases we want to compile a circuit in which some constants can be changed. This can be done with a tunable integer or a tunable boolean. A name prefixed by a `@` is a tunable integer, while one prefixed by a `#` is a tunable boolean.

Note that not all these constants are accepted by every compiler version. An error message will be produced when necessary[3].

A.2 Executing Commands

Most commands have a text form, which starts with an exclamation mark, and a short form, which is usually made up of symbols, both can be used interchangeably. When the command uses a λ-term, the current λ-term is used (usually the last one that was entered).

- `!alias`, or `=` defines an alias that represents the current λ-term. The command must be followed by a valid name for the alias. Each time an alias appears as a free variable in an typed λ-term, it is be expanded to the stored value. Expansion does not change terms that have already been entered.
- `!disp` or `?` displays the current λ-term.
- `!eval` or `-` evaluates the current λ-term.
- `!vars` or `??` displays all the aliases.
- `!types` or `'` displays the types of all the sub-terms of the current λ-term.
- `!ltypes` or `''` displays the types of all the sub-terms of the current λ-term if it is linear, and reports an error otherwise.
- `!lcompile` or `"` generates Esterel code for a linear λ-term.

[3]The error will appear on standard output and not on standard error because Caml seems to swallow output to standard error at times.

- `!icompile` or `"""` generates Esterel code for a lambda term through interaction combinators. The code uses two stacks.
- `!sicompile` or `""` does the same as `!icompile` except that the interaction net goes through $\gamma - \gamma$ and $\delta - \delta$ reduction, as well as $\gamma - \delta$ exchange as seen in section 3.7.2.
- `!ticompile` or `"""""` generates Esterel code for a λ-term by using an interaction combinator type system as in section 3.7.3. The resulting circuit only uses one stack. Not all terms can be compiled this way.
- `!drawicomb` or `???` draws the current λ-term compiled into interaction combinators. Code that is custom-written in *termdraw.ml* then interactively evaluates the term.
- `!gammaproc` or `+` followed by an integer n generates a gamma processor with 2^{n+1} gamma agents, 2^n adders and 2^n integers, then steps through its operation.
- `!fastgammaproc` or `++` followed by an integer n generates a gamma processor with 2^{n+1} gamma agents, 2^n adders and 2^n integers, then runs it at full speed.
- `!exit` or *EOF* exits the program.

A.3 Compiling the Esterel code

For each type of compilation, an shell dump will be given to show how to get the generated code working. For detailed instructions on how to use the Esterel software, refer to [Ber].

When entering terms, you will find that not all encodings support all constants of the lambda calculus. Some don't support pairs, some don't support integers but have tunable integers, some are the other way round. Don't worry, it is always possible to do simple increments and decrements on integers.

To compile to a file other than test.strl, you have to duplicate the test.h header file for whatever name you choose.

A.3.1 Compiling a Linear λ-term

First start *lambda2hard* with output redirected to a file.

```
gassend@bataplan ~/source $ ./lambda2hard  > test.strl
```

Now enter a lambda expression and compile it.

```
(fn x -> x) (fn x -> inc x) 5
"
!exit
```

Compile the resulting Esterel file.

```
gassend@bataplan ~/source $ esterel -main BODY -B test -simul test.strl \
lib.strl
```

Note that without the -B option, Esterel will behave in a well documented but undesirable way, since it will try to compile unused library modules, as well as the main module.

Finally you can simulate the compiled term in *xes* the Esterel simulator.

```
gassend@bataplan ~/source $ xes test.c stack.c unit.c
### --- Linking ...
### cc -o /tmp/xes2695178175744.exe test.c stack.c unit.c
/share/svrlix/users/demons/gassend/usr/esterel/lib/libxes.a
/share/svrlix/users/demons/gassend/usr/esterel/tcltk/lib/libtk8.2.a
/share/svrlix/users/demons/gassend/usr/esterel/tcltk/lib/libtcl8.2.a
-L/usr/X11R6/lib -lX11 -ldl -lm
### --- Done
### /tmp/xes2695178175744.exe 2>/dev/tty
```

At this point, the Esterel simulator starts. A token can be started by sending a signal on one of the inputs.

A.3.2 Compiling into Typed Interaction Combinators

Here we go through the same basic steps, but the term can be more complex (though not all terms can be compiled this way).

```
gassend@bataplan ~/source \$ ./lambda2hard > test.strl
(fn x y z -> y (x (y z))) inc dec @tuneint
"""
!exit
gassend@bataplan ~/source $ esterel -main BODY -B test -simul test.strl \
lib.strl
gassend@bataplan ~/source $ xes test.c stack.c unit.c
### --- Linking ...
### cc -o /tmp/xes3948179141524.exe test.c stack.c unit.c
/share/svrlix/users/demons/gassend/usr/esterel/lib/libxes.a
/share/svrlix/users/demons/gassend/usr/esterel/tcltk/lib/libtk8.2.a
/share/svrlix/users/demons/gassend/usr/esterel/tcltk/lib/libtcl8.2.a
-L/usr/X11R6/lib -lX11 -ldl -lm
### --- Done
### /tmp/xes3948179141524.exe 2>/dev/tty
```

This time the simulator needs to be told the value of `@tuneint` before it can start. Don't be surprised if this is much slower than the one-hot encoding from the previous example.

A.3.3 Compiling into Two-Stack Interaction Combinators

There are two commands to compile into two-stack interaction combinators, depending on whether the optimization steps should be run. I will only illustrate one.

Everything is as before except that a different library is used[4].

[4] In some cases constructiveness analysis might has to be turned on with the *-causal* option when running Esterel.

```
gassend@bataplan ~/source $ ./lambda2hard > test.strl
Y (fn f n m -> ifte (iszero n) m (f (dec n) (inc m)))
= add
add 1 1
"""
!exit
gassend@bataplan ~/source $ esterel -main BODY -B test -simul test.strl \
simplelib.strl
gassend@bataplan ~/source $ xes test.c stack.c unit.c
### --- Linking ...
### cc -o /tmp/xes4114179144806.exe test.c stack.c unit.c
/share/svrlix/users/demons/gassend/usr/esterel/lib/libxes.a
/share/svrlix/users/demons/gassend/usr/esterel/tcltk/lib/libtk8.2.a
/share/svrlix/users/demons/gassend/usr/esterel/tcltk/lib/libtcl8.2.a
-L/usr/X11R6/lib -lX11 -ldl -lm
```

You can expect this example to take many steps (and a lot of clicking) to run. An alternative way of executing it is :

```
gassend@cygne ~/source $ esterel -main BODY -B test test.strl
simplelib.strl
gassend@cygne ~/source $ gcc mymain.c stack.c unit.c -o mymain
gassend@cygne ~/source $ ./mymain
IntegerRegister : 1
BooleanRegister : 0
IntegerRegister : 1
IntegerRegister : 0
BooleanRegister : 1
IntegerRegister : 1
IntegerRegister : 2
done delta : 52, gamma : 1768
```

With this method execution is continuous, with whatever information is requested in *mymain.c* being displayed. The result 2, is correctly in the register at the end of execution. Of course *mymain.c* can be customized at will, tunable constants can be set, different information can be displayed, and so on.

Note that *mymain.c* includes *temp.c*, so changes will have to be made if the Esterel output file has a different name.

Appendix B

Overview of the Source Code

Here is a brief description of each file in the source directory. Because of a problem with my tab stop size, they will probably have to be reindented before use, unless a tab size of 2 is used.

Makefile The Makefile.

RCS Contains older revisions of the files. A few odds and ends are in the RCS directory that do not appear here.

constants.ml Defines the constants of the λ-calculus.

debug.ml Used for debugging purposes.

electronic.ml Produces Esterel code from a Caml data structure.

error.ml Displays error messages.

gammaprocessor.ml The first version of the Gamma Processor. Is now obsolete.

gammaprocessor2.ml Generates and simulates a Gamma Processor.

icomb.ml Compiles a λ-term into interaction combinators while adding some typing information (γ information is stored in the type).

icombsimple.ml Compiles a λ-term into interaction combinators without any typing information.

icombsimplify.ml Simplifies an interaction net by doing $\gamma - \gamma$ and $\delta - \delta$ reduction, as well as $\gamma - \delta$ exchange as seen in section 3.7.2.

inet2circuit.ml Compiles a typed interaction net into a circuit with one stack.

inet2circuitsimple.ml Compiles an untyped interaction net into a circuit with two stacks.

inettype.ml Functions for manipulating the types used by *icombsimple.ml* and *inet2circuitsimple.ml*

lambda.ml Defines the λ-expression types and a simple evaluator.

lexer.mll The lexer for typed input.

lib.strl A library of used by some of the circuits compiled into Esterel.

lin2circuit.ml Converts a linear lambda term into an Esterel circuit.

main.ml Calls up the parser after preparing a few odds and ends.

mymain.c Can be used to simulate two stack Esterel circuits without being limited by the step by step functioning of the Esterel simulator.

naming.ml Functions to help choose names.

parser.mly Parses command lines and calls the apropriate module.

parsercode.ml Various bits of code needed by the parser.

parserinterface.ml Functions to call the parser.

simplelib.strl Library for the two stack interaction combinator circuits.

stack.c Implements a bit stack for use from Esterel.

stack.h Header for *stack.c*

symbol.ml Takes care of storing symbol names in a hash table.

symbol.mli Interface of *symbol.ml*

termdraw.ml Draws and interprets a term in the two stack interaction combinator encoding.

test.h test2.h Includes the proper headers for code generated by Esterel.

test.lambda Various terms that can be used as examples.

testandset.ml A type, which holds a value that can only be set once.

types.ml Functions for typing λ-expressions, and checking that a term is linear.

unit.c Implements the unit type for Esterel code. It is very unfortunate that the pure signals in Esterel do not share a common syntax with other signals. The unit type makes signal sintax uniform.

unit.h Header for *unit.c*

Appendix C

Interaction Combinator Graphical Interface

Since the full details of how the geometry of interaction works on λ-terms compiled into interaction combinators are not yet fully understood, the interface that I wrote to study them is a collection of classes to draw the term, and a set of hooks, which hopefully should be sufficient to easily insert breakpoints at key structural points in the net. The following sections are designed to give an overview of the source in `termdraw.ml`, with the main focus on using the hooks.

C.1 Outline

Once the program is running, the net is represented by pairs of `netend` classes. A netend is the end of a wire. Conventionnaly, an end also represents a direction of flow through a wire. Each `netend` class knows how to draw itself, how to move a token to the next netend, and what hooks need to be called when it is executed.

Execution takes place in the `main_loop` function, which is a very likely candidate for user customization. The sample main loop simply steps through execution, and should serve as a model for other main loops. A `machine_state` class keeps track of the machine's data[1].

Before being passed to the main loop, the net and the graphic window are initialized by `init_draw` from a λ-expression. The construction of the net is done through the function `full_term`, and prcoeeds in two phases. In the first phase, the term is recursively encoded using `termblock` classes, which immediately calculate the display room that they will need. In the second pass, `termblock` classes are told their actual coordinates, which allow them to draw themselves and store line coordinates in `netend`s. There is a subclass of `termblock` for each type of structure in the λ expression.

[1] The γ and δ stacks and the current constant value.

C.2 Setting Hooks

Each `netend` has a list of functions that must be called each time it is executed. I call them hooks. Hooks can be added from the main loop, to insert a breakpoint at the current net, for example. But in my opinion, the most interesting use of hooks is to do a specified action each time a particular type of wire is crossed. Such hooks are added during net construction. By default they do nothing, but they can easily be changed to keep track of invariants.

The hooks that I have defined apply to the following events :

- Entering or leaving an auxiliary port of an abstracter or applier agent.
- The data register changing.
- Entering or leaing copy mode.[2]

They can be configured by changing the following lines of `termdraw.ml`, which are located around line 200.

```
let abstractparami () = ()
let abstractparamo () = ()
let abstractvaluei () = ()
let abstractvalueo () = ()
let abstractcopy1i () = ()
let abstractcopy1o () = ()
let abstractcopy2i () = ()
let abstractcopy2o () = ()
let abstractorigi () = ()
let abstractorigo () = ()

let applyparami () = ()
let applyparamo () = ()
let applyvaluei () = ()
let applyvalueo () = ()
let applycopy1i () = ()
let applycopy1o () = ()
let applycopy2i () = ()
let applycopy2o () = ()
let applyorigi () = ()
let applyorigo () = ()

let deltai () = ()
let deltao () = ()

let copystart n () = ()
let copyend n () = ()

let datachangedhook d = ()
```

With very little work in `netend#step`, it would be possible for these functions to have the `machine_state` or current `netend` classes passed to them.

[2]This notion is defined in figure 3.12.

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